Journal of Financial Markets 7 (2004) 187–206

Journal of FINANCIAL MARKETS

www.elsevier.com/locate/econbase

Manipulation in market order models☆

Archishman Chakraborty[a], Bilge Yılmaz[b,*]

[a] *Department of Economics and Finance, Baruch College, CUNY, USA*
[b] *Finance Department, The Wharton School, University of Pennsylvania, USA*

Abstract

We analyze a dynamic market order model similar to Kyle (Econometrica 53 (1985) 1315). We show that when the market faces uncertainty about the existence of the insider in the market, the equilibrium outcome changes in a significant way. In particular, the insider manipulates (i.e., trades in the wrong direction and undertakes short term losses) in every equilibrium, given a long enough horizon, and independently of the precise nature of noise trading in the market.

JEL classification: D82; G12

Keywords: Insider trading; Manipulation

---

1. Introduction

In a seminal paper, Kyle (1985) investigates the optimal trading of an informed insider in sequential auctions where his trading affects the equilibrium price. In the unique linear equilibrium, the informed trader strategically chooses to trade less aggressively than he would in a competitive situation. Nevertheless, the equilibrium does not involve manipulation of prices. To quote from Kyle (1985), the unique linear equilibrium of the model "rules out a situation in which the insider can make unbounded profits by first destabilizing prices with unprofitable trades made at the

---

☆We are indebted to Franklin Allen, Simon Gervais, Faruk Gül, Timothy VanZandt, and an anonymous referee for helpful comments and criticism. The second author thanks the Rodney L. White Center for Financial Research at the University of Pennsylvania for support. The usual disclaimer applies.

*Corresponding author. Tel.: +1-215-898-1163; fax: +1-215-898-6200.
E-mail address: yilmaz@wharton.upenn.edu (B. Yılmaz).

1386-4181/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/j.finmar.2003.11.001

*n*th auction, then recouping the losses and much more with profitable trades at future auctions'' (p. 1323).

In this paper, we consider precisely this kind of manipulative strategic trading by the informed insider. As in Kyle (1985) we have three kinds of traders: an informed trader, noise traders and competitive market makers. In every period all traders first submit their orders. The competitive market-maker adds up all the orders and chooses a price at which to execute them, observing only the aggregate order flow. As a result, in every period any individual trader faces uncertainty about the price at which the order will be executed. In the context of such a market order model we find that every equilibrium involves manipulation by the informed trader if the horizon is long enough. In other words, in every equilibrium, not only do informed traders want to hide their trades by trading less aggressively, but also find it in their interest to confuse other market participants by trading in the ''wrong'' direction for short-term losses but long-term profits. For example, an insider who knows that the prospects of a certain asset are not good might actually start buying the asset in order to drive its price up and then sell it without its price falling too fast. This stark difference from Kyle's well-known result obtains because our market order model differs from his in two principal ways.

First, we assume that the market does not know with certainty that an informed insider exists in the market. Specifically, we assume that if the insider does not exist then he is replaced by a noise trader whose (exogenous) trading strategy is uncorrelated with the value of the asset. This additional uncertainty gives an incentive to the informed trader to manipulate so that he can mislead the market makers about the existence of an informed trader in the market. If the informed insider does exist then we assume, like Kyle, that the market does not know the nature of his information.

While we assume that with positive probability the informed trader is replaced by a noise trader, we put very little restriction on the strategy of the noise trader apart from the assumption that it has full support. We allow this strategy to be history and price dependent. Furthermore, we obtain our result on the necessity of manipulation for long horizons by deriving a bound that is independent of the precise specification of the noise trader strategy. Consequently, our result is robust to the possibility of endogenizing the noise.

The second difference from Kyle's model is that we assume that the possible trade sizes are finite, whereas Kyle assumes that the possible trade sizes that can be submitted in any period by any trader lie in a continuum. The assumption of finitely many trade sizes eases the construction of the extensive form trading game that we utilize. Because (apart from this finiteness assumption) we do not put any assumptions on the distribution of noise traders' trades, any model with a continuum of trade sizes can be arbitrarily closely approximated with our finite model, by choosing a grid that is sufficiently fine.

The fact that manipulation is both possible and profitable in our model is in line with what policy-makers generally appear to believe.[1] In the context of our model

---

[1] See Securities Exchange Act of 1934, Section 9.

manipulation does not involve "buying low and selling high". Rather, by undertaking unprofitable trades early on, the informed trader is able to add more noise to the price formation process. This enables him to trade profitably in the direction of his information subsequently without an undue adverse impact on prices. In effect, the informed trader exploits the market makers' uncertainty about his presence to create his own liquidity or noise.

Other papers that explore the issue of manipulation include Jarrow (1992), Allen and Gale (1992), Allen and Gorton (1992), Kumar and Seppi (1992), Kyle (1984). The present paper departs from this literature by considering an explicit strategic model of trading where manipulation by an informed trader necessarily occurs and is strictly profitable, even in the absence of a secondary derivative security market.[2] Brunnermeier (2000) also considers strategic trading by an informed trader in a Kyle setting, although manipulation (in the sense of undertaking unprofitable short-term trades) does not occur in his model. Chakraborty and Yılmaz (2004) prove a result similar to that of the present paper but in a Glosten-Milgrom setting. Fishman and Hagerty (1995), John and Narayanan (1997) and Huddart et al. (2001) show that manipulative trading may occur due to the presence of mandatory disclosure laws.

In Section 2 we introduce the model and prove that the insider will manipulate in every equilibrium if his information is long-lived. Section 3 concludes, and the appendix contains the proof of the main result.

2. Model

2.1. The trading game

We consider a market for one risky asset with one riskless asset whose gross rate of return is normalized to 1. The long-term return or the fundamental value of the risky asset, v, is not known to all participants in the market. In particular, we assume $v \in V = \{0, 1\}$, with the prior probability that $v = 1$ equal to $\pi \in (0, 1)$.

There is one dynamic trader in the market who trades repeatedly. The private information or type of the informed trader is denoted by $\theta \in \Theta = \{\mathbf{0}, \mathbf{1}, \mathbf{N}\}$. When $\theta = \mathbf{0}$, the dynamic trader is informed and knows that the value of the asset is $v = 0$. When $\theta = \mathbf{1}$, the dynamic trader is informed and knows that the value of the asset is $v = 1$. When $\theta = \mathbf{N}$, the dynamic trader is a "noise" trader and his trading is driven by exogenous motives. The existence of this last type of trader is meant to capture the notion that the market faces an uncertainty regarding the existence of a dynamic trader who trades on the basis of information about the fundamentals. We allow the trading strategy of type $\theta = \mathbf{N}$ of the dynamic trader to depend on the histories of trades as well as prices.

---

[2] Bernhardt et al. (2002) also point out the possibility of manipulation by an informed trader in a derivative securities market. See Chakraborty and Yılmaz (2004) for a detailed review of the literature.

We suppose that the prior distribution of θ is specified by

$$\Pr[\theta = \mathbf{1}, | v = 1] = \Pr[\theta = \mathbf{0} \,|\, v = 0] \equiv \mu \in (0,1), \tag{1}$$

while

$$\Pr[\theta = \mathbf{N} \,|\, v = 1] = \Pr[\theta = \mathbf{N} \,|\, v = 0] \equiv 1 - \mu. \tag{2}$$

We suppose that the informed dynamic trader is risk-neutral and maximizes the expected undiscounted sum of his per-period profits. We will specify the strategies and payoffs of the dynamic trader more formally below.

We consider a market order model of sequential trading with discrete trade sizes where market makers observe the aggregate market order in each period and post a price to execute the order. The market makers do not know the realization of v or θ. The market works as follows: the dynamic trader submits an order, $x_t \in X$, in period $t = 1, 2, \ldots,$ and noise traders submit an order $y_t \in Y$. The market makers observe the aggregate order

$$h_t = x_t + y_t \tag{3}$$

and set prices.

We now formally set up our trading game by defining strategies, payoffs and beliefs for both the dynamic trader and the market maker.

We suppose that for integers $\bar{x},\ \bar{y} \geqslant 1$,

$$X = \{-\bar{x}, -\bar{x}+1, \ldots, -1, 0, 1, \ldots, \bar{x}-1, \bar{x}\} \tag{4}$$

and

$$Y = \{-\bar{y}, -\bar{y}+1, \ldots, -1, 0, 1, \ldots, \bar{y}-1, \bar{y}\}. \tag{5}$$

The set of possible market orders is

$$H = \{-\bar{x}-\bar{y}, \ldots, -1, 0, 1, \ldots, \bar{x}+\bar{y}\}. \tag{6}$$

Trading occurs for T periods before all private information is publicly revealed.

We suppose that the distribution of trades y_t that is submitted in each period t, in addition to the dynamic trader's trades x_t, is identical and independent across periods. Specifically let $g(y)$ be the probability that $y_t = y$, with $G(y) = \sum_{y'=-\bar{y}}^{y} g(y)$ being the cumulative distribution function. We assume that $g(y) > 0$ for all $y \in Y$ and $g(y) = 0$ for $y \notin Y$.

Let h, x, y denote the generic elements of H, X, and Y, respectively. Denote by h^t, the t-period history of market orders $\{h_1, h_2, \ldots, h_t\}$, and denote by H^t the set of t-period histories. Let $H^0 = \{h^0\}$, be the null history set. Let $\mathbf{h}^{t'}(h^t)$ denote the first t' elements of the history h^t and let $\mathbf{h}_{t'}(h^t)$ the t'th element, $t \geqslant t'$. For $t \geqslant t'$, let $H^t(h^{t'}) = \{h^t \in H^t \,|\, \mathbf{h}^{t'}(h^t) = h^{t'}\}$. Let $\bar{H} = \bigcup_{t=0}^{\infty} H^t$.

Let x^t denote the sequence of trades submitted by the dynamic trader for the first t periods, X^t denoting the set of such trades. Let $X^0 = \{x^0\}$, the set of the null sequence of trades. Let $\mathbf{x}^{t'}(x^t)$ be the first t' elements of x^t and let $\mathbf{x}_{t'}(x^t)$ the t'th element, $t \geqslant t'$. For $t \geqslant t'$, let $X^t(x^{t'}) = \{x^t \in X^t \,|\, \mathbf{x}^{t'}(x^t) = x^{t'}\}$.

The timing structure of the T-period trading game described above is as follows:

1. Nature chooses v and θ. The dynamic trader observes θ.

2. In successive periods indexed by $t = 1, \ldots, T$, the dynamic trader submits an order x_t, and the market maker, having observed the history h^{t-1} and $h_t = x_t + y_t$, chooses a price.
3. In period $T + 1$, the realization of v is publicly disclosed.

For the dynamic trader and the market maker, a trading strategy specifies (a possibly random) action after every observed history of trades and past prices. However, given an observed history, the past prices are irrelevant for the future, for both the dynamic trader and the market maker. As a result, in our definition of strategies, we omit their dependence on past prices.

Therefore, a strategy for the dynamic trader specifies a probability distribution over possible trade sizes in X for each possible history of past trades, market orders and prices. Let $\Delta(X)$ be the space of probability distributions over X and, for $t = 1, 2, \ldots$, define the sets

$$\Sigma_t = \{\sigma_t \mid \sigma_t : X^{t-1} \times H^{t-1} \to \Delta(X)\} \tag{7}$$

and let

$$\Sigma = \times_{t=1}^{\infty} \Sigma_t. \tag{8}$$

Any strategy σ for the dynamic trader is a sequence of functions $\{\sigma_t\}_t$, an element of Σ. For any t, and $x^{t-1} \in X^{t-1}$, $h^{t-1} \in H^{t-1}$, we will denote by $\sigma_t(x \mid x^{t-1}, h^{t-1})$ the probability that σ assigns to action $x \in X$, given x^{t-1}, h^{t-1}. Σ is obviously a convex set and so is $\Sigma \times \Sigma$. We will denote by $\sigma_\theta = \{\sigma_{\theta t}\}_t$ the strategy of type $\theta \in \Theta$ of the dynamic trader.

We suppose that, when the dynamic trader is not informed ($\theta = \mathbf{N}$), he buys, sells, and does not trade with probability bounded away from zero for all histories:

(N) There exists $c > 0$ s.t. $\sigma_{\mathbf{N}t}(x \mid x^{t-1}, h^{t-1}) > c$ for all x, x^{t-1}, h^{t-1} and $t \geqslant 1$.

Note that except for condition (N), we do not put any restriction on the strategy of type $\theta = \mathbf{N}$: it could depend on the past history of trades as well as on current prices. Furthermore, our result on the necessity of manipulation for long horizons will not depend on the precise specification of the strategy $\sigma_{\mathbf{N}}$. Specifically, we obtain a bound $\mathbf{T}^*$, *independent of* $\sigma_{\mathbf{N}}$, such that if the horizon of the game $T > \mathbf{T}^*$ then every equilibrium would involve manipulation by the informed types of the dynamic trader. As a result, our result on the necessity of manipulation is robust to the possibility of $\sigma_{\mathbf{N}}$ being endogenized, as long as $\sigma_{\mathbf{N}}$ satisfies condition (N).

In any period t, after observing a history of trades h^{t-1} and the current order flow h_t, the market maker chooses a price, $p(\{h^{t-1}, h_t\}) \in [0, 1]$. We can therefore write the strategy of the market maker as a function $p : \bar{H} \to [0, 1]$. Let P denote the set of strategies for the market maker.[3]

Note that the strategy of the market maker as well as that of the dynamic trader have been defined to apply to finite as well as infinite horizons, even though we

---

[3] Note that this specification of the market maker's strategy implies that the market maker also chooses a price $p(h^0)$ after the null history h^0.

restrict attention to finite horizon trading games in this paper. This formulation allows us to define strategy sets in a manner that is independent of the trading horizon, a convenience we utilize in the proof of our result. We incorporate the finite horizon of the trading game by defining payoff functions and our equilibrium notion suitably.

A strategy $\sigma \in \Sigma$ and the noise trader distribution g induce a probability distribution over the possible histories. For $t \geqslant t' \geqslant 0$, let $q(x^t, h^t \mid x^{t'}, h^{t'}, \sigma)$ be the probability with which the sequence $\{x^t,\ h^t\}$ is generated by σ given that $x^{t'}, h^{t'}$ has occurred. Similarly, let $q(h^t \mid h^{t'}, \sigma)$ be the probability with which h^t is generated by σ given that $h^{t'}$ has occurred. That is,

$$q(x^t, h^t \mid x^{t'}, h^{t'}, \sigma) = \prod_{t''=t'+1}^{t} \sigma_{t''}(\mathbf{x}_{t''}(x^t) \mid \mathbf{x}^{t''-1}(x^t), \mathbf{h}^{t''-1}(h^t))\, g(\mathbf{h}_{t''}(h^t) - \mathbf{x}_{t''}(x^t)) \quad (9)$$

if $x^t \in X^t(x^{t'})$ and $h^t \in H^t(h^{t'}),\ t > t'$; equal to 1 if $x^t = x^{t'}$ and $h^t = h^{t'}$; and is equal to 0 otherwise. Similarly,

$$q(h^t \mid h^{t'}, \sigma) = \sum_{x^t \in X^t} \prod_{t''=t'+1}^{t} \sigma_{t''}(\mathbf{x}_{t''}(x^t) \mid \mathbf{x}^{t''-1}(x^t), \mathbf{h}^{t''-1}(h^t))\, g(\mathbf{h}_{t''}(h^t) - \mathbf{x}_{t''}(x^t)) \quad (10)$$

if $h^t \in H^t(h^{t'}),\ t > t'$; equal to 1 if $h^t = h^{t'}$; and is equal to 0 otherwise. We will suppress notation and denote by $q(x^t, h^t \mid \sigma)$ the probability that $\{x^t, h^t\}$ is generated by σ, given the null sequence $\{x^0, h^0\}$ and denote by $q(h^t \mid \sigma)$ the probability that h^t is generated by σ, given the null history h^0.

We now define the market maker's conditional beliefs on a state of the world given an observed history h^t. For any $h^t \in \bar{H}$, we denote by $\tilde{\pi}(v \mid h^t)$, the belief, conditional on h^t, that the market maker assigns to the event that the value of the asset is $v \in V$. Let $\tilde{\pi}$ denote the collection $\{\tilde{\pi}(v \mid h^t)\}_{v,h^t}$. Similarly, let $\tilde{\mu}(\theta \mid v, h^t)$ denote the belief that the dynamic trader is of type θ given v and h^t with $\tilde{\mu}$ denoting the associated collection. Finally, for $t \geqslant t',\ h^t, h^{t'} \in \bar{H}$, let $\tilde{q}(h^t \mid v, \theta, h^{t'})$ be the belief that history h^t will be generated given $h^{t'},\ \theta$ and v, with $\tilde{q}$ denoting the associated collection.[4] Given $\{\tilde{\pi}, \tilde{\mu}, \tilde{q}\}$ we can, using the usual properties of conditional probabilities, derive all possible joint, marginal and conditional distributions on $V \times \Theta \times H^t$ for all t, that represent the market maker's system of beliefs given an observed history.[5] We will use the symbol Q as the general representation of the market maker's beliefs on $V \times \Theta \times H^t$, which includes $\{\tilde{\pi}, \tilde{\mu}, \tilde{q}\}$ as well as all other conditional beliefs derived from them. Q is determined in equilibrium by the strategy profile $\{\sigma_\theta\}_{\theta \in \Theta}$ used by the dynamic trader and the priors μ and π.

We now explicitly define an objective function to be maximized by the market maker. Our approach is outcome equivalent to market efficiency assumption in Kyle (1985). Instead of modeling two market makers with an explicit Bertrand auction we define one market maker and an appropriate payoff function for each terminal node,

---

[4] Notice that even though the market maker only observes the order flow h^t, we have specified a complete system of *conditional beliefs* for the market maker.

[5] Note that for $t \geqslant t'$, each $h^{t'}$ is identified with a unique subset of $H^t(h^{t'})$ of H^t. Thus when the market maker observes $h^{t'}$ he knows that $H^t(h^{t'}) \subset H^t$ has occurred.

in such a way that the market maker sets prices equal to the expected value of the asset given his information, for every observed history h^t and for all periods up to period T.

We suppose the market maker's one-period payoff from any trade at any price $\hat{p}$ is $-(v-\hat{p})^2$ and that he maximizes the expected undiscounted sum of his per-period profits.[6] Notice that the payoff of the market maker is the negative of the distance between v and the price he sets $\hat{p}$. This will imply that in equilibrium, the market maker will choose a price equal to the expected value of v conditional on the observed history, for every history.

For $t \in \{0, \dots, T-1\}$, and any h^t, we define the expected continuation payoff of the market maker, $M_{t,T}$, from date t till date T given h^t, as follows:

$$\begin{aligned} M_{t,T}(p, Q \mid h^t) = & -\sum_{v} \tilde{\pi}(v \mid h^t)(v - p(h^t))^2 \\ & + \sum_{h \in H} Q[\{h^t, h\} \mid h^t] M_{t+1,T}(p, Q \mid \{h^t, h\}), \end{aligned} \tag{11}$$

where

$$Q[\{h^t, h\} \mid h^t] \equiv \sum_{\theta} \sum_{v} \tilde{q}(h^T \mid v, \theta, h^t)\tilde{\mu}(\theta \mid v, h^t)\tilde{\pi}(v \mid h^t)$$

and

$$M_{T,T}(p, Q \mid h^T) = -\sum_{v} \tilde{\pi}(v \mid h^T)(v - p(h^T))^2. \tag{12}$$

The one period payoff from any trade of size x at any price $\hat{p}$ for type $\theta \in \{\mathbf{0}, \mathbf{1}\}$ of the dynamic trader is $(\theta - \hat{p})x$. We suppose that the dynamic trader maximizes the expected undiscounted sum of his per-period profits. For $t \in \{1, \dots, T-1\}$, the expected continuation payoffs, from date t till date T, given x^{t-1}, h^{t-1}, of type $\theta \in \{\mathbf{0}, \mathbf{1}\}$ of the informed trader, from any strategy σ_θ and prices p, is denoted by $U_{1,T}(\theta, \sigma_\theta, p \mid x^{t-1}, h^{t-1})$ and defined to be equal to

$$\begin{aligned} & \sum_{x} \sigma_{\theta t}(x \mid x^{t-1}, h^{t-1}) \sum_{h} g(h - x)[(\theta - p(\{h^{t-1}, h\}))x \\ & \quad + U_{t+1,T}(\theta, \sigma_\theta, p \mid \{x^{t-1}, x\}, \{h^{t-1}, h\})] \end{aligned} \tag{13}$$

with

$$\begin{aligned} U_{T,T}(\theta, \sigma_\theta, p \mid x^{T-1}, h^{T-1}) = & \sum_{x} \sigma_{\theta t}(x \mid x^{T-1}, h^{T-1}) \sum_{h} g(h - x) \\ & \times (\theta - p(\{h^{T-1}, h\}))x. \end{aligned} \tag{14}$$

Note that the expected payoff to type θ from date 1 equals $U_{1,T}(\theta, \sigma_\theta, p \mid x^0, h^0)$. As it causes no confusion we will drop the dependence on the null sequence $\{x^0, h^0\}$ and denote the expected payoff at the beginning of the game by $U_{1,T}(\theta, \sigma_\theta, p)$.

---

[6] The price $\hat{p}$ here is a number in [0, 1], not to be confused with the market-maker's strategy p, which is a *function* choosing a price for every history.

2.2. Equilibrium

Since we have exogenous noise trading in the model, it follows that the set of games that we look at are defined by this strategy $\sigma_{\mathbf{N}}$, the constant c in condition (N), by the distribution g, by the number T of trading periods, and the parameters μ and π. Accordingly, we will denote our T-period trading game by $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$. The equilibrium notion that we consider for such a game Γ is that of (weak) perfect Bayesian equilibrium.

Definition 1. An equilibrium for $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ is a tuple of strategies and beliefs $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ such that the following hold.

1. The informed types' strategies are sequentially rational given p: for all $\theta \in \{\mathbf{0}, \mathbf{1}\}$, for all $t = 1, \ldots, T$, for all x^{t-1}, h^{t-1}, σ_θ maximizes type θ's expected continuation payoff $U_{t,T}(\theta, \sigma_\theta, p \mid x^{t-1}, h^{t-1})$.
2. The market-maker's strategy p is sequentially rational given Q: for all $t = 0, \ldots, T$ and for all h^t, the market maker's strategy p maximizes the market-maker's continuation payoff $M_{t,T}(p, Q \mid h^t)$.
3. For all $t = 0, \ldots, T$, and h^t, the beliefs Q are derived from the strategies $\{\sigma_\theta\}_{\theta \in \Theta}$ via Bayes' Rule whenever possible.

From condition 3 in the definition of equilibrium, since $\sigma_{\mathbf{N}}$ satisfies (N), we must have for all h^t, $t = 1, \ldots, T$,

$$\tilde{\pi}(1 \mid h^t) = \frac{\{\mu q(h^t \mid \sigma_{\mathbf{1}}) + (1-\mu) q(h^t \mid \sigma_{\mathbf{N}})\}\pi}{(1-\mu) q(h^t \mid \sigma_{\mathbf{N}}) + \mu q(h^t \mid \sigma_{\mathbf{1}})\pi + \mu q(h^t \mid \sigma_{\mathbf{0}})(1-\pi)}. \tag{15}$$

Further, $\tilde{\pi}(1 \mid h^t) \in (0, 1)$ for all h^t.

From condition 2 in the definition of equilibrium and (12) it is immediate that the price $p(h^T)$ after any history h^T will be set equal to $\tilde{\pi}(1 \mid h^T)$. Thus, from (11) we must have by backward induction that:

$$p(h^t) = \tilde{\pi}(1 \mid h^t) \quad \text{for all } h^t, \ t = 1, \ldots, T. \tag{16}$$

Therefore, our specification of market maker payoffs and our notion of equilibrium generate the competitive outcome. In any period, the equilibrium price will equal the expected value of the asset conditional on observing the history of trades up to and including that period.

2.3. Manipulative strategies

We now define our notion of manipulation. An informed trader, when selecting his trade in any period, must balance the short-term profit from the trade with the long-term effect his trade has on the beliefs of the market and hence on future profits. We say that a strategy is manipulative if it involves the informed trader undertaking a trade in any period that yields a *strictly negative* short-term profit.

If such a strategy is used in equilibrium, then it must be to manipulate the beliefs of the market regarding his private information, which will enable him to recoup the short-term losses (and more) in the future.

Definition 2. Given prices p, a strategy $\sigma \in \Sigma$ is non-manipulative for type $\theta \in \{\mathbf{0}, \mathbf{1}\}$, if, for all $t \geqslant 1$ and x^{t-1}, h^{t-1}, x

$$\sigma_t(x \mid x^{t-1}, h^{t-1}) > 0 \;\Rightarrow\; \sum_h g(h-x)[\theta - p(\{h^{t-1}, h\})]\, x \geqslant 0.$$

Otherwise σ is manipulative for type θ.

Therefore, a strategy is non-manipulative for type θ if type θ earns non-negative expected profits whenever an order x is submitted. Note that this definition of non-manipulative strategies implies a joint restriction on the strategy of the informed trader and the strategy of the market maker (i.e., the prices chosen). For any strategy p of the market maker let $\Sigma_\theta^{nm}(p)$ be the set of non-manipulative strategies for type $\theta \in \{\mathbf{0}, \mathbf{1}\}$, with

$$\Sigma^{nm}(p) = \Sigma_{\mathbf{0}}^{nm}(p) \times \Sigma_{\mathbf{1}}^{nm}(p). \tag{17}$$

Note from (15) and (16) that in *any* equilibrium of the T-period game, the price $p(h^t)$ must lie in the interval $(0, 1)$, for any history h^t, $t = 1, \ldots, T$. As a result, for any such equilibrium prices, the non-manipulative strategies of type $\theta = 1$ cannot put positive probability on any $x<0$, and those for type $\theta = \mathbf{0}$ cannot put positive probability on any $x > 0$, for any x^{t-1}, h^{t-1}, $t = 1, \ldots, T$.

For such prices, the condition for a strategy to qualify as non-manipulative is quite weak. For type $\theta = \mathbf{1}$, not buying with probability 1, even if the price is lower than its expected value, is not considered manipulative: he can choose not to trade even if it is profitable to trade.[7] Similarly, the myopic or one-shot optimal strategy of always buying with good news and always selling with bad news is non-manipulative, but it is not the only non-manipulative strategy.[8] Finally, note that manipulative strategies are not necessarily mixed strategies and that mixed strategies are not necessarily manipulative.[9]

2.4. The main result

Theorem 1. *For any μ, π, g, c, there exists $\mathbf{T}^*(\mu, \pi, g, c)$, such that for all $T \geqslant \mathbf{T}^*(\mu, \pi, g, c)$ and any $\sigma_{\mathbf{N}}$ satisfying (N), every equilibrium of the game $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ involves manipulation on the path of play.*

Proof. See the Appendix. □

---

[7] In other words, choosing the timing of trading is not considered, by this definition, to be manipulative.

[8] This would be part of the equilibrium if the dynamic trader were replaced with many identical one-period traders. Our main result is thus stronger than the claim that the equilibrium strategy of the dynamic informed trader will be different from that generated by many such short-lived traders.

[9] For $\theta = 1$, the pure strategy of selling just once and buying after that is manipulative, and the mixed strategy of buying and not trading every period with positive probability is non-manipulative.

Considering the fact that this game is not finite, in the appendix we first prove that an equilibrium exists for all T.[10] Here we focus on the intuition underlying why the equilibrium must involve manipulation for long enough horizons.

Suppose there is an equilibrium in non-manipulative strategies. Then the informed trader of type $\theta = \mathbf{1}$ is not supposed to sell with positive probability in any period in this candidate non-manipulative equilibrium, as the expected price at which he can sell is always less than 1. But suppose type $\theta = \mathbf{1}$ deviates and goes on selling in every period. Given his beliefs, the market maker will observe an order inconsistent with type $\theta = \mathbf{1}$ trading in the market, with arbitrarily large probability, if type $\theta = \mathbf{1}$ sells long enough. Since all histories can be generated by type $\theta = \mathbf{N}$, the market maker will then put zero weight on type $\theta = \mathbf{1}$ trading in the market, and the price will be less than or equal to π, in every period to follow, no matter what history occurs. The trader can thus buy any amount at a price less than or equal to π in every period for a profit of $1 - \pi$. In contrast, we show in the Appendix that in any candidate non-manipulative equilibrium, for T large enough, per-period profit is bounded away from $1 - \pi$. Thus, for T large enough, type $\theta = \mathbf{1}$ has a profitable deviation from his candidate equilibrium strategy. In fact, since the strategy sets are compact, we can find a $\mathbf{T}^*$ that is independent of the strategies used (including $\sigma_{\mathbf{N}}$) such that if $T \geqslant \mathbf{T}^*$, every equilibrium involves manipulation.

In the argument above, we could conclude that the market maker will necessarily observe an order inconsistent with type $\theta = \mathbf{1}$'s strategy, *only* because the candidate equilibrium involves no manipulation.[11] The profitable deviation strategy from this candidate equilibrium, and the market maker's response to this deviation, do not necessarily occur in equilibrium.

3. Conclusion

The primary contribution of this paper is to show that in Kyle (1985) type of models with one insider trading repeatedly, as long as the number of periods is large enough, the equilibrium will involve a manipulative trading strategy of the insider as long as (i) the market faces uncertainty about the existence of the insider and (ii) the number of periods is large. The uncertainty about the existence of the insider gives an incentive to the insider to manipulate in order to signal that he is not trading on any information. When sufficiently many periods of trading are left, this enables him to recoup his initial losses and make further profits.

---

[10] Note that market maker in each period can choose any price in the $[0, 1]$ interval, resulting in an infinite strategy space.

[11] Clearly, in an equilibrium where type $\theta = 1$ manipulates observing a net order flow less than $-\bar{y}$ will not make the market maker put weight on type $\theta = 1$.

Appendix A. Proof of the theorem

We endow Σ_t with the topology generated by the metric $d_t : \Sigma_t \times \Sigma_t \to \mathbb{R}$ defined by

$$d_t(\sigma_t, \sigma'_t) \equiv \max_{h^{t-1} \in H^{t-1}, x^{t-1} \in X^{t-1}} \max_{x \in X} |\sigma_t(x \mid x^{t-1}, h^{t-1}) - \sigma'_t(x \mid x^{t-1}, h^{t-1})(x)|. \tag{A.1}$$

Note that for each t, Σ_t is a compact subset of a finite-dimensional Euclidean space. The metric $D : \Sigma \times \Sigma \to \mathbb{R}$ defined by

$$D(\sigma, \sigma') = \sup_{t \geqslant 1} \left\{ \frac{d_t(\sigma_t, \sigma'_t)}{t} \right\}, \tag{A.2}$$

induces the product topology on Σ. By Tychonov's theorem, Σ is compact relative to the product topology. Endow $\Sigma \times \Sigma$ with the product topology generated by (A.2). Since Σ is compact, so is $\Sigma \times \Sigma$.

We prove the theorem in three steps. Lemma 1 establishes that an equilibrium exists for the game $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$. In Lemma 2, we obtain a bound on the equilibrium expected payoffs in any candidate equilibrium involving no manipulation. In Lemma 3, we show that this implies that there is an upper bound on the length of the game for which the equilibria can involve no manipulation.

A.1. Existence of equilibrium

Lemma 2. *An equilibrium exists for the game* $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$.

Proof. Fixing a $\sigma_{\mathbf{N}}$, define the best-response function of the market maker $p^* : \Sigma \times \Sigma \to P$, where $p^*(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}})(h^t)$ is the price after history h^t according to $p^*(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in P$, and is given by the right-hand side of (15). Note from (15) and (16), that if $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ is an equilibrium, we must have, for all h^t,

$$p(h^t) = p^*(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}})(h^t). \tag{A.3}$$

We now define a best-reply correspondence for each type $\theta \in \{\mathbf{0}, \mathbf{1}\}$ and show that it has a fixed point. Then we show that from the fixed point of this correspondence we can construct equilibrium in the sense of Definition 1.

For $1 \leqslant t \leqslant T$, define the correspondences $\xi^t : \Sigma \times \Sigma \rightrightarrows \Sigma \times \Sigma$ as follows:

$$\xi^t_{\mathbf{1}}(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) = \{\sigma \in \Sigma \mid \sigma \in \arg\max_{\hat{\sigma}} U_{t',T}(\mathbf{1}, \hat{\sigma}, p^*(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \mid x^{t'-1}, h^{t'-1}) \forall x^{t'} - 1, h^{t'-1}, \\ \times\ t \leqslant t' \leqslant T\}.$$

$$\xi^t_{\mathbf{0}}(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) = \{\sigma \in \Sigma \mid \sigma \in \arg\max_{\hat{\sigma}} U_{t',T}(\mathbf{0}, \hat{\sigma}, p^*(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \mid x^{t'-1}, h^{t'-1}) \forall x^{t'} - 1, h^{t'-1}, \\ \times\ t \leqslant t' \leqslant T\}.$$

We want to show that ξ^1 has a fixed point. Standard arguments involving induction on t establish that ξ^1 is non-empty convex-valued correspondence. Furthermore, using the continuity properties of the payoff functions defined (13) and

(14) it is straightforward to verify that ξ^1 has a closed graph.[12] Since $\Sigma \times \Sigma$ is compact and convex, and ξ^1 is a non-empty valued, convex-valued correspondence with a closed graph on $\Sigma \times \Sigma$, it has a fixed point, by the Kakutani-Fan-Glicksberg theorem (see, e.g., Moore (1999)). Let $(\sigma^*_{\mathbf{0}}, \sigma^*_{\mathbf{1}})$ be a fixed-point of ξ^1.

Let $\tilde{\pi}$ and $\sigma^*_{\mathbf{0}}, \sigma^*_{\mathbf{1}}$ satisfy (15) for all h^t. Further, let

$$\tilde{\mu}(\mathbf{0} \mid v, h^t) = \begin{cases} \dfrac{q(h^t \mid \sigma^*_{\mathbf{0}})\mu}{q(h^t \mid \sigma^*_{\mathbf{0}})\mu + q(h^t \mid \sigma_{\mathbf{N}})(1-\mu)} & \text{if } v = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{A.4}$$

$$\tilde{\mu}(\mathbf{1} \mid v, h^t) = \begin{cases} \dfrac{q(h^t \mid \sigma^*_{\mathbf{1}})\mu}{q(h^t \mid \sigma^*_{\mathbf{1}})\mu + q(h^t \mid \sigma_{\mathbf{N}})(1-\mu)} & \text{if } v = 1, \\ 0 & \text{otherwise} \end{cases} \tag{A.5}$$

for all h^t, and let

$$\tilde{q}(h^t \mid v, \theta, h^{t'}) = q(h^t \mid h^{t'}, \sigma^*_{\theta}) \tag{A.6}$$

for all v, θ, h^t and $h^{t'}$, $t \geqslant t'$. This defines a system of beliefs Q of the market maker, which satisfy Bayes' Rule with respect to the strategies $\sigma^*_{\mathbf{0}}, \sigma^*_{\mathbf{1}}$ and the priors μ and π, whenever Bayes' Rule can be applied. Then it is immediate that $\{\sigma^*_{\mathbf{0}}, \sigma^*_{\mathbf{1}}, p^*(\sigma^*_{\mathbf{0}}, \sigma^*_{\mathbf{1}}), Q\}$ is an equilibrium according to Definition 1. This concludes the proof of existence. □

A.2. Non-existence of non-manipulative equilibria

Suppose that $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ is an equilibrium of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$, with $(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in \Sigma^{nm}(p)$. Note first that since $\sigma_{\mathbf{N}}$ satisfies (N) and $\mu < 1$, we must have $p(h^t) \in (0,1)$ for all h^t, $t = 1, \ldots, T$. Since $\sigma_{\mathbf{1}} \in \Sigma^{nm}_{\mathbf{1}}(p)$ and $\sigma_{\mathbf{0}} \in \Sigma^{nm}_{\mathbf{0}}(p)$ we must have by Definition 2,

$$\sigma_{\mathbf{0}t}(x \mid x^{t-1}, h^{t-1}) = 0 \tag{A.7}$$

and

$$\sigma_{\mathbf{1}t}(x' \mid x^{t-1}, h^{t-1}) = 0 \tag{A.8}$$

for all $x > 0$, $x' < 0$, $x^{t-1}, h^{t-1}, t = 1, \ldots, T$.

Thus, if $\theta = \mathbf{0}$, no $h_t > \bar{y}$ can be observed on the equilibrium path; and, if $\theta = \mathbf{1}$, no $h_t < -\bar{y}$ can be observed on the equilibrium path, given that $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ is an equilibrium of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ with $(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in \Sigma^{nm}(p)$. We will use these facts repeatedly in the next proof.

The next lemma establishes an upper bound on payoffs in any candidate equilibrium using non-manipulative strategies, where the upper bound for each type $\theta \in \{\mathbf{0}, \mathbf{1}\}$ depends only on the strategy used by that type and not on $\sigma_{\mathbf{N}}$ or the strategy used by the other type.

---

[12] For details see Chakraborty and Yılmaz (2000).

Lemma 3. *If* $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ *is an equilibrium of* $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$, *with* $(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in \Sigma^{nm}(p)$, *then there exists* $k_\theta(\sigma_\theta) \in (0, 1)$, $\theta \in \{\mathbf{0}, \mathbf{1}\}$ *such that*

$$U_{1,T}(\mathbf{0}, \sigma_{\mathbf{0}}, p) < \left[k_{\mathbf{0}}(\sigma_{\mathbf{0}})\pi T + \frac{1}{g(-\bar{y})}\right]\bar{x}. \tag{A.9}$$

$$U_{1,T}(\mathbf{1}, \sigma_{\mathbf{1}}, p) < \left[k_{\mathbf{1}}(\sigma_{\mathbf{1}})(1-\pi) T + \frac{1}{g(\bar{y})}\right]\bar{x}, \tag{A.10}$$

Further, $k_\theta(\sigma_\theta)$ *does not depend on* $\sigma_{\mathbf{N}}$ *or* $\sigma_{\theta'}$ *for* $\theta' \neq \theta$.

Proof. Suppose $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ is an equilibrium of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ with $(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in \Sigma^{nm}(p)$.

We derive below the second inequality (A.9), for type $\theta = \mathbf{0}$. The proof for type $\theta = \mathbf{1}$ is entirely analogous.

We proceed in steps. In step 1 we show that for histories which cannot be reached by $\theta = \mathbf{1}$'s strategy, the expected future price given $\theta = \mathbf{0}$ is trading is less than the current price. In step 2, we use this to put an upper bound on $\theta = \mathbf{0}$'s expected continuation payoff for all such histories. In step 3, we put an upper bound on $\theta = \mathbf{0}$'s expected continuation payoffs for *any* history and date. In step 4, we combine steps 2 and 3 to establish (A.9).

Step 1: For $t \in \{1, \dots, T\}$, define

$$A_t^{\mathbf{0}} = \{h^t \mid \mathbf{h}_{t'}(h^t) < -\bar{y} \text{ for some } t' \leqslant t\}.$$

Note from (16) and (A.8) we must have

$$p(h^t) \leqslant \pi \text{ for all } h^t \in A_t^{\mathbf{0}}. \tag{A.11}$$

Further, for all $h^t \in A_t^{\mathbf{0}}$,

$$\sum_{h^{t+1} \in H^{t+1}(h^t)} q(h^{t+1} \mid h^t, \sigma_{\mathbf{0}}) p(h^{t+1}) < p(h^t). \tag{A.12}$$

To see why this holds, note from (16) and the usual properties of conditional probabilities that for all h^t and each $h^{t+1} \in H^{t+1}(h^t)$, the price $p(h^{t+1})$ equals

$$\frac{q(h^{t+1} \mid h^t, \sigma_{\mathbf{N}}) p(h^t)}{q(h^{t+1} \mid h^t, \sigma_{\mathbf{N}}) p(h^t) + \{q(h^{t+1} \mid h^t, \sigma_{\mathbf{N}}) \tilde{\mu}(\mathbf{N} \mid 0, h^t) + q(h^{t+1} \mid h^t, \sigma_{\mathbf{0}}) \tilde{\mu}(\mathbf{0} \mid 0, h^t)\}(1 - p(h^t))}, \tag{A.13}$$

as, for $h^t \in A_t^{\mathbf{0}}$,

$$\tilde{\mu}(\mathbf{N} \mid h^t, 1) = 1 = \tilde{\mu}(\mathbf{N} \mid 0, h^t) + \tilde{\mu}(\mathbf{0} \mid 0, h^t).$$

Using (A.13), it is easily seen that the expression on the left-hand side of (A.12), as a function of $\{q(h^{t+1} \mid h^t, \sigma_{\mathbf{0}})\}_{h^{t+1} \in H^{t+1}(h^t)}$, is strictly concave. Further, for fixed $\{q(h^{t+1} \mid h^t, \sigma_{\mathbf{N}})\}_{h^{t+1} \in H^{t+1}(h^t)}$, $p(h^t)$ and $\tilde{\mu}(\mathbf{N}|0, h^t)$, if we choose $\{q(h^{t+1} \mid h^t, \sigma_{\mathbf{0}})\}_{h^{t+1} \in H^{t+1}(h^t)}$ to maximize the expression on the left-hand side of (A.12), subject only to the constraint that $\{q(h^{t+1} \mid h^t, \sigma_{\mathbf{0}})\}_{h^{t+1} \in H^{t+1}(h^t)}$ is a probability distribution on $H^{t+1}(h^t)$, then the maximum value of the objective function is equal to $p(h^t)$, which is achieved if and only if $q(h^{t+1} \mid h^t, \sigma_{\mathbf{0}}) = q(h^{t+1} \mid h^t, \sigma_{\mathbf{N}})$ for all $h^{t+1} \in H^{t+1}(h^t)$. But this is impossible by (A.7) as $\sigma_{\mathbf{N}}$ satisfies condition (N) and $\sigma_{\mathbf{0}} \in \Sigma^{nm}(p)$, so that we obtain the inequality in (A.12).

Step 2: We show that, for all $1 \leqslant t \leqslant T$, $h^{t-1} \in A^{\mathbf{0}}_{t-1}$, and all x^{t-1},

$$U_{t,T}(\mathbf{0}, \sigma_{\mathbf{0}}, p \mid x^{t-1}, h^{t-1}) < p(h^{t-1})(T - t + 1)\bar{x}. \tag{A.14}$$

We proceed by induction. For $t = T$, and any $h^{T-1} \in A^{\mathbf{0}}_{T-1}$ and any x^{T-1},

$$\begin{aligned} U_{T,T}(\mathbf{0}, \sigma_{\mathbf{0}}, p \mid x^{T-1}, h^{T-1}) &\leqslant \sum_{x} \sum_{h} q(\{x^{T-1}, x\}, \{h^{T-1}, h\} \mid x^{T-1}, h^{T-1}, \sigma_{\mathbf{0}}) \\ &\quad \times p(\{h^{T-1}, h\})\bar{x} \\ &\leqslant \sum_{h} q(\{h^{T-1}, h\} \mid h^{T-1}, \sigma_{\mathbf{0}}) p(h^{T})\bar{x} \\ &< p(h^{T-1})\bar{x}, \end{aligned}$$

from (9), (10), (14) and Step 1.

Suppose (A.14) holds for all $t \geqslant t' + 1$. For $t = t'$ and any $h^{t'-1} \in A^{\mathbf{0}}_{t'-1}$, all $x^{t'-1}$, $U_{t',T}(\mathbf{0}, \sigma_{\mathbf{0}}, p \mid x^{t'-1}, h^{t'-1})$ is less than or equal to

$$\begin{aligned} &\sum_{x} \sum_{h} q(\{x^{t'-1}, x\}, \{h^{t'-1}, h\} \mid x^{t'-1}, h^{t'-1}, \sigma_{\mathbf{0}})[p(\{h^{t'-1}, h\})\bar{x} \\ &+ U_{t'+1,T}(\mathbf{0}, \sigma_{\mathbf{0}}, p \mid \{x^{t'-1}, x\}, \{h^{t'-1}, h\})], \end{aligned}$$

which in turn is strictly less than

$$\begin{aligned} &\sum_{x} \sum_{h} q(\{x^{t'-1}, x\}, \{h^{t'-1}, h\} \mid x^{t'-1}, h^{t'-1}, \sigma_{\mathbf{0}}) p(\{h^{t'-1}, h\})(T - t' + 1)\bar{x} \\ &\quad < p(h^{t'-1})(T - t' + 1)\bar{x}, \end{aligned}$$

where in the last line above we have made use of our inductive hypothesis, (9), (10) and Step 1. This establishes (A.14).

Step 3: We now show by induction that for all $1 \leqslant t \leqslant T$, all h^{t-1} and x^{t-1}

$$U_{t,T}(\mathbf{0}, \sigma_{\mathbf{0}}, p \mid x^{t-1}, h^{t-1}) < \left[\pi(T - t + 1) + \frac{1 - g(-\bar{y})}{g(-\bar{y})}\right]\bar{x}. \tag{A.15}$$

For $t = T$, any h^{T-1}, x^{T-1}, by (A.7),

$$U_{T,T}(\mathbf{0}, \sigma_{\mathbf{0}}, p \mid x^{T-1}, h^{T-1}) = -\sum_{x \leqslant 0} \sigma_{\mathbf{0}T}(x \mid x^{T-1}, h^{T-1}) \sum_{h} g(h - x) p(\{h^{T-1}, h\})x.$$

For all h^{T-1} and h, we know that $p(\{h^{T-1}, h\}) < 1$. From (A.7), $\sigma_{\mathbf{0}T}(0 \mid x^{T-1}, h^{T-1})$ for all x^{T-1} and h^{T-1}. Further, from (A.8) and (A.11), $p(h^{T}) \leqslant \pi$ for all $h^{T} \in A^{\mathbf{0}}_{T}$.

Thus,

$$\begin{aligned}
&U_{T,T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid x^{T-1},h^{T-1})\\
&\quad< \sum_{x<0} \sigma_{\mathbf{0}T}(x \mid x^{T-1},h^{T-1})[G(-\bar{y}-x-1)\pi+(1-G(-\bar{y}-x-1))]\bar{x}\\
&\quad\leqslant [G(-\bar{y})\pi+1-G(-\bar{y})]\bar{x}\\
&\quad< \left[\pi+\frac{1-g(-\bar{y})}{g(-\bar{y})}\right]\bar{x},
\end{aligned}$$

the right-hand side of (A.15) for $t=T$, as $g(-\bar{y})=G(-\bar{y})$.

Suppose that (A.15) holds for all $t\geqslant t'+1$. Then, for $t=t'$ and any $x^{t'-1},h^{t'-1}$, since $\sigma_{\mathbf{0}}$ is a best-response after $x^{t'-1},h^{t'-1}$, the expected continuation payoff $U_{t',T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid x^{t'-1},h^{t'-1})$ is equal to

$$\sum_{h} g(h-x')[p(\{h^{t'-1},h\})(-x')+U_{t'+1,T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid \{x^{t'-1},x'\},\{h^{t'-1},h\})],$$

the expected continuation payoff from some action $x'\leqslant 0$ in the support of $\sigma_{\mathbf{0}}$ after $x^{t'-1},h^{t'-1}$. If $x'=0$, then by the inductive hypothesis,

$$\begin{aligned}
U_{t',T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid x^{t'-1},h^{t'-1}) &= \sum_{h} g(h)[U_{t'+1,T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid \{x^{t'-1},0\},\{h^{t'-1},h\})]\\
&< \left[\pi(T-t')+\frac{1-g(-\bar{y})}{g(-\bar{y})}\right]\bar{x}\\
&< \left[\pi(T-t'+1)+\frac{1-g(-\bar{y})}{g(-\bar{y})}\right]\bar{x}.
\end{aligned}$$

So suppose $x'<0$. Recall from (A.8) and (A.11) that $p(h^t)\leqslant\pi$ for all $t\geqslant t'$ and $h^t\in A_t^{\mathbf{0}}$ and that $p(h^t)<1$ for all h^t. Then, using our inductive hypothesis, $U_{t',T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid x^{t'-1},h^{t'-1})$ is less than

$$\begin{aligned}
&G(-\bar{y}-x'-1)[\pi+\pi(T-t')]\bar{x}\\
&\quad+(1-G(-\bar{y}-x'-1))\left[1+\pi(T-t')+\frac{1-g(-\bar{y})}{g(-\bar{y})}\right]\bar{x}\\
&\quad\leqslant \pi(T-t')\bar{x}\left[g(-\bar{y})\pi+(1-g(-\bar{y}))\left(1+\frac{1-g(-\bar{y})}{g(-\bar{y})}\right)\right]\bar{x}\\
&\quad< \left[\pi(T-t'+1)+\frac{1-g(-\bar{y})}{g(-\bar{y})}\right]\bar{x}.
\end{aligned}$$

This establishes (A.15).

Step 4: For any $t\geqslant 1$ let $x_0^{t-1}\in X^{t-1}$ be a sequence of $t-1$ successive orders of size 0. Let

$$q(h^{t-1} \mid x_0^{t-1}) \equiv \prod_{t'=1}^{t-1} g(\mathbf{h}_{t'}(h^{t-1})),$$

be the probability that h^{t-1} is generated, given x_0^{t-1} has been played. Let

$$H_g^{t-1}=\{h^{t-1} \mid q(h^{t-1} \mid x_0^{t-1})>0\}. \tag{A.16}$$

Pick some $\varepsilon \in (0,1)$ and let

$$t_1(\sigma_{\mathbf{0}},\varepsilon) \equiv \min\{t \mid \exists h^{t-1} \in H_g^{t-1} \text{ s.t. } \sigma_{\mathbf{0}t}(0 \mid x_0^{t-1}, h^{t-1}) \leqslant 1-\varepsilon\}. \tag{A.17}$$

That is, $t_1(\sigma_{\mathbf{0}},\varepsilon)$ is the first period in which $\sigma_{\mathbf{0}}$ attaches probability at most $1-\varepsilon$ to $x=0$, for some history which is generated with positive probability given that only orders of size 0 have been submitted before that. Since $\sigma_{\mathbf{0}}$ is an equilibrium strategy for some T, there must exists such a $t_1(\sigma_{\mathbf{0}},\varepsilon)$, as otherwise the expected profits of type $\theta = \mathbf{0}$ is equal to 0, the expected profits from not trading ever. For notational brevity, we will omit the dependence of $t_1(\sigma_{\mathbf{0}},\varepsilon)$ on $\sigma_{\mathbf{0}}$ and ε in what follows, as long as it causes no confusion. Let $h_*^{t_1-1} \in H_g^{t_1-1}$ be such that

$$\sigma_{\mathbf{0}t_1}(0 \mid h_*^{t_1-1}, x_0^{t_1-1}) \leqslant 1-\varepsilon. \tag{A.18}$$

Since $\sigma_{\mathbf{0}}$ is an equilibrium strategy, from the definition of t_1, the expected profits of type $\theta = \mathbf{0}$ is equal to the expected profits from not trading till period $t_1 - 1$ and then trading according to $\sigma_{\mathbf{0}}$. That is,

$$\begin{aligned} U_{1,T}(\mathbf{0},\sigma_{\mathbf{0}},p) &= q(h_*^{t_1-1} \mid x_0^{t_1-1}) U_{t_1,T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid x_0^{t_1-1}, h_*^{t_1-1}) \\ &\quad + \sum_{h^{t_1-1} \neq h_*^{t_1-1}} q(h^{t_1-1} \mid x_0^{t_1-1}) U_{t_1,T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid x_0^{t_1-1}, h^{t_1-1}). \end{aligned} \tag{A.19}$$

From step 3, for $h^{t_1-1} \neq h_*^{t_1-1}$,

$$U_{t_1,T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid x_0^{t_1-1}, h^{t_1-1}) < \left[\pi(T-t_1+1) + \frac{1-g(-\bar{y})}{g(-\bar{y})}\right]\bar{x}. \tag{A.20}$$

Further, from (A.18), there exists $x^* < 0$ with $\sigma_{\mathbf{0}t_1}(x_0^{t_1-1}, h_*^{t_1-1})(x^*) > 0$ such that $U_{t_1,T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid x_0^{t_1-1}, h_*^{t_1-1})$ is equal to

$$\sum_h g(h - x^*)[p(\{h_*^{t_1-1}, h\})(-x^*) + U_{t_{1+1},T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid \{x_0^{t_1-1}, x^*\}, \{h_*^{t_1-1}, h\})].$$

The terms in the sum above can be broken up into two groups—those for which $h < -\bar{y}$ and those for which $h \geqslant -\bar{y}$.

For $h \geqslant -\bar{y}$, since prices are bounded above by 1, each such term can be bounded above using step 3 as follows:

$$\begin{aligned} & p(\{h_*^{t_1-1}, h\})(-x^*) + U_{t_{1+1},T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid \{x_0^{t_1-1}, x^*\}, \{h_*^{t_1-1}, h\}) \\ &\quad < \left[p(\{h_*^{t_1-1}, h\}) + \pi(T-t_1) + \frac{1-g(-\bar{y})}{g(-\bar{y})}\right]\bar{x} \\ &\quad < \left[\pi(T-t_1+1) + \frac{1}{g(-\bar{y})}\right]\bar{x}. \end{aligned} \tag{A.21}$$

On the other hand, for $h < -\bar{y}$, $\{h_*^{t_1-1}, h\} \in A_{t_1}^{\mathbf{0}}$, so that by step 2,

$$\begin{aligned} & p(\{h_*^{t_1-1}, h\})(-x^*) + U_{t_{1+1},T}(\mathbf{0},\sigma_{\mathbf{0}},p \mid \{x_0^{t_1-1}, x^*\}, \{h_*^{t_1-1}, h\}) \\ &\quad < [p(\{h_*^{t_1-1}, h\})(T-t_1+1)]\bar{x}. \end{aligned} \tag{A.22}$$

Pick $\delta > 0$ with $\delta < \min[\varepsilon/t_1, (1-\varepsilon)/t_1]$ and let $\underline{g} = \min_{y \in Y}\{g(y)\}$. Let

$$K_{\mathbf{0}}(\sigma_{\mathbf{0}},\varepsilon,\delta) \equiv (1-\varepsilon-\delta t_1)^{t_1-1}(\varepsilon-\delta t_1)\underline{g}^{t_1} > 0 \tag{A.23}$$

and note that for all $h < -\bar{y}$, since $\{h_*^{t_1-1}, h\} \in A_{t_1}^{\mathbf{0}}$,

$$\begin{aligned} p(\{h_*^{t_1-1}, h\}) &= \frac{q(\{h_*^{t_1-1}, h\} \mid \sigma_{\mathbf{N}})(1-\mu)\pi}{q(\{h_*^{t_1-1}, h\} \mid \sigma_{\mathbf{N}})(1-\mu) + q(\{h_*^{t_1-1}, h\} \mid \sigma_{\mathbf{0}})\mu(1-\pi)} \\ &< \frac{(1-\mu)\pi}{(1-\mu) + q(\{h_*^{t_1-1}, h\} \mid \sigma_{\mathbf{0}})\mu(1-\pi)} \\ &< \frac{(1-\mu)\pi}{(1-\mu) + K_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta)\mu(1-\pi)} \\ &\equiv \bar{p}(\sigma_{\mathbf{0}}, \varepsilon, \delta) < \pi, \end{aligned} \tag{A.24}$$

where in the second line we have used the fact that $q(\{h_*^{t_1-1}, h\} \mid \sigma_{\mathbf{N}}) < 1$ and $q(\{h_*^{t_1-1}, h\} \mid \sigma_{\mathbf{0}}) > 0$, and in the third line we have used (A.17) and (A.18) to conclude that

$$q(\{h_*^{t_1-1}, h\} \mid \sigma_{\mathbf{0}}) \geqslant (1-\varepsilon)^{t_1-1}\varepsilon \underline{g}^{t_1} > K_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta).$$

Combining (A.20)–(A.24) in (A.19) we have

$$\begin{aligned} U_{1,T}(\mathbf{0}, \sigma_{\mathbf{0}}, p) &< \left[k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta)\pi(T - t_1 + 1) + \frac{1}{g(-\bar{y})}\right]\bar{x} \\ &\leqslant \left[k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta)\pi T + \frac{1}{g(-\bar{y})}\right]\bar{x}, \end{aligned}$$

where

$$\begin{aligned} k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta) \equiv{}& q(h_*^{t_1-1} \mid x_0^{t_1-1})\left[g(-\bar{y})\frac{\bar{p}(\sigma_{\mathbf{0}}, \varepsilon, \delta)}{\pi} + 1 - g(-\bar{y})\right] + 1 \\ &- q(h_*^{t_1-1} \mid x_0^{t_1-1}). \end{aligned} \tag{A.25}$$

Since $q(h_*^{t_1-1} \mid x_0^{t_1-1}) > 0$, we must have $k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta) \in (0, 1)$. Note that $k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta)$ does not depend on $\sigma_{\mathbf{1}}$ or $\sigma_{\mathbf{N}}$. Moreover, $k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta)$ depends on $\sigma_{\mathbf{0}}$ (and ε and δ) because both $\bar{p}(\sigma_{\mathbf{0}}, \varepsilon, \delta)$ and $t_1(\sigma_{\mathbf{0}}, \varepsilon)$ do. Since $\varepsilon \in (0, 1)$ was arbitrarily chosen and, given ε and $\sigma_{\mathbf{0}}$, so was δ, this establishes (A.9). This completes the proof of the lemma. □

For $t = 1, 2, \ldots$, let

$$\Sigma_{\mathbf{0}t}^{nm} = \{\sigma_t \in \Sigma_t \mid \sigma_t(x \mid x^{t-1}, h^{t-1}) = 0 \text{ for all } x > 0, x^{t-1}, h^{t-1}\}, \tag{A.26}$$

$$\Sigma_{\mathbf{1}t}^{nm} = \{\sigma_t \in \Sigma_t \mid \sigma_t(x \mid x^{t-1}, h^{t-1}) = 0 \text{ for all } x < 0, x^{t-1}, h^{t-1}\} \tag{A.27}$$

and for $\theta \in \{\mathbf{0}, \mathbf{1}\}$ let

$$\Sigma_\theta^{nm} = \times_t \Sigma_{\theta t}^{nm} \subset \Sigma, \tag{A.28}$$

and

$$\Sigma^{nm} = \Sigma_{\mathbf{0}}^{nm} \times \Sigma_{\mathbf{1}}^{nm}. \tag{A.29}$$

Note that, for $\theta \in \{\mathbf{0}, \mathbf{1}\}$, $\Sigma_{\theta t}^{nm}$ is a closed and so compact subset of the compact set Σ_t. By Tychonov's theorem, Σ_θ^{nm} is a compact set relative to the product topology introduced on Σ in (A.2).

Notice that if $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ is an equilibrium of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ (for some $\sigma_{\mathbf{N}}$) then, for $\theta \in \{\mathbf{0}, \mathbf{1}\}$,

$$\Sigma_\theta^{nm} \subset \Sigma_\theta^{nm}(p) = \left\{ \mathop{\times}_{t=1}^{T} \Sigma_{\theta t}^{nm} \right\} \times \left\{ \mathop{\times}_{t \geqslant T+1} \Sigma_t \right\}.$$

This is because (15) and (16) imply that $p(h^t) \in (0, 1)$, for all h^t, $t = 0, \ldots, T$, with the price being unrestricted for dates after date T, so that (A.6) and (A.7) put a restriction on the set of non-manipulative strategies only up to date T. However, if $\sigma_\theta \in \Sigma_\theta^{nm}(p)$ for all $\theta \in \{\mathbf{0}, \mathbf{1}\}$, then there exists another equilibrium $\{\sigma'_{\mathbf{0}}, \sigma'_{\mathbf{1}}, p, Q'\}$, with $\sigma'_\theta \in \Sigma_\theta^{nm}$ for all θ with

$$\sigma'_{\theta t} = \sigma_{\theta t} \in \Sigma_{\theta t}^{nm} \quad \text{for all } t = 1, \ldots, T,$$

and Q' derived from $\{\sigma'_{\mathbf{0}}, \sigma'_{\mathbf{1}}\}$ whenever possible.

To complete the proof of the theorem we want to show that there exists $\mathbf{T}^*(\mu, \pi, g, c)$ such that for $T \geqslant \mathbf{T}^*(\mu, \pi, g, c)$ and any $\sigma_{\mathbf{N}}$ satisfying (N), there does not exist an equilibrium $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ with $(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in \Sigma^{nm}(p)$. In view of the previous paragraph, it suffices then to prove the next lemma.

Lemma 4. *There exists* $\mathbf{T}^*(\mu, \pi, g, c)$ *such that for any* $\sigma_{\mathbf{N}}$ *satisfying* (*N*), *if* $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ *is an equilibrium of* $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ *with* $\sigma_\theta \in \Sigma_\theta^{nm}$ *for all* $\theta \in \{\mathbf{0}, \mathbf{1}\}$, *then* $T < \mathbf{T}^*(\mu, \pi, g, c)$.

Proof. Using the previous lemma we show first that if $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ is an equilibrium of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ with $(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in \Sigma^{nm}$ then $T < \mathbf{T}(\sigma_{\mathbf{0}})$ for some $\mathbf{T}(\sigma_{\mathbf{0}})$ which depends on $\sigma_{\mathbf{0}}$ but not on $\sigma_{\mathbf{1}}$ or on $\sigma_{\mathbf{N}}$. We then use the compactness of $\Sigma_{\mathbf{0}}^{nm}$ to obtain the upper bound $\mathbf{T}^*(\mu, \pi, g, c)$ that does not depend on $\sigma_{\mathbf{0}}$ to complete the proof.

Step 1: Pick any equilibrium $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ with $(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in \Sigma^{nm}$.

Consider the following deviation strategy for type $\theta = \mathbf{0}$. Suppose he trades an amount $\bar{x}$ for at most $t(\sigma_{\mathbf{0}})$ periods and then trades $-\bar{x}$ till period T as soon as $h_t > \bar{y}$ for some $t \leqslant t(\sigma_{\mathbf{0}})$. Assume that if $h_t \leqslant \bar{y}$ for all $t \leqslant t(\sigma_{\mathbf{0}})$, then his deviation strategy prescribes not trading in periods $t > t(\sigma_{\mathbf{0}})$. Note that since $\sigma_{\mathbf{0}} \in \Sigma_{\mathbf{0}}^{nm}$, all future prices are at least as high as π after any period in which $h_t > \bar{y}$.

Type $\theta = \mathbf{0}$'s profit $u_{\mathbf{0}}^d$, from this deviation strategy can be bounded below as follows:[13]

$$u_{\mathbf{0}}^d > [-t(\sigma_{\mathbf{0}}) + (1 - (G(\bar{y} - \bar{x}))^{t(\sigma_{\mathbf{0}})})\pi T]\bar{x}.$$

Let $t(\sigma_{\mathbf{0}})$ be large enough so that

$$1 - (G(\bar{y} - \bar{x}))^{t(\sigma_{\mathbf{0}})} > k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta),$$

---

[13] If $\bar{x} > 2\bar{y}$, then $G(-\bar{y}) > G(\bar{y} - \bar{x}) = 0$. If $2\bar{y} \geqslant \bar{x}$, then $G(\bar{y} - \bar{x}) > 0$.

where $k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta) \in (0,1)$ is defined in (A.25). Then there exists $\mathbf{T}(\sigma_{\mathbf{0}}) \in \mathbb{R}$ such that for all $T \geqslant \mathbf{T}(\sigma_{\mathbf{0}})$:

$$\begin{aligned} U_{1,T}(\mathbf{0}, \sigma_{\mathbf{0}}, p) &< \left[k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta)\pi T + \frac{1}{g(-\bar{y})} \right] \bar{x} \\ &< [-t(\sigma_{\mathbf{0}}) + (1 - (G(\bar{y} - \bar{x}))^{t(\sigma_{\mathbf{0}})})\pi T]\bar{x} \\ &< u_{\mathbf{0}}^{d}. \end{aligned}$$

Note that $\mathbf{T}(\sigma_{\mathbf{0}})$ does not depend on $\sigma_{\mathbf{N}}$ or $\sigma_{\mathbf{1}}$ as $k_{\mathbf{0}}(\sigma_{\mathbf{0}})$ does not. Thus if $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ is an equilibrium of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ with $(\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}) \in \Sigma^{nm}$ then $T < \mathbf{T}(\sigma_{\mathbf{0}})$ for some $\mathbf{T}(\sigma_{\mathbf{0}})$ which depends on $\sigma_{\mathbf{0}}$ but not on $\sigma_{\mathbf{1}}$ or $\sigma_{\mathbf{N}}$.

Step 2: For a fixed $\sigma_{\mathbf{0}}$, and δ as in step 4 of the proof of the last lemma, let $B(\delta, \sigma_{\mathbf{0}}) \subset \Sigma_{\mathbf{0}}^{nm}$ be the open δ-ball around $\sigma_{\mathbf{0}}$. We now show that for all $\sigma'_{\mathbf{0}} \in B(\delta, \sigma_{\mathbf{0}})$, and any $\sigma'_{\mathbf{1}} \in \Sigma_{\mathbf{1}}^{nm}$, if $\{\sigma'_{\mathbf{0}}, \sigma'_{\mathbf{1}}, p', Q'\}$ is an equilibrium of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T')$ then $T' < \mathbf{T}(\sigma_{\mathbf{0}})$.

Since $\sigma'_{\mathbf{0}} \in B(\delta, \sigma_{\mathbf{0}})$, for all $t < t_1(\sigma_{\mathbf{0}}, \varepsilon)$ and all $h^{t-1} \in H_g^{t-1}$,

$$\sigma'_{\mathbf{0}t}(0 \mid x_0^{t-1}, h^{t-1}) > 1 - \varepsilon - \delta t > 1 - \varepsilon - \delta t_1(\sigma_{\mathbf{0}}, \varepsilon) > 0,$$

where x_0^{t-1} is the sequence of $t-1$ orders of size 0, as before and $t_1(\sigma_{\mathbf{0}}, \varepsilon)$ and H_g^{t-1} have been defined in (A.17) and (A.16), respectively. Further,

$$\sigma'_t(0 \mid x_0^{t_1(\sigma_{\mathbf{0}},\varepsilon)-1}, h_*^{t_1(\sigma_{\mathbf{0}},\varepsilon)-1}) \leqslant 1 - \varepsilon + \delta t_1(\sigma_{\mathbf{0}}, \varepsilon) < 1,$$

where $h_*^{t_1(\sigma_0,\varepsilon)-1}$ is defined by (A.18).

Since $\sigma'_{\mathbf{0}}$ is part of an equilibrium, the expected profits $U_{1,T'}(\mathbf{0}, \sigma'_{\mathbf{0}}, p')$ is equal to that from submitting orders of size 0 till period $t_1(\sigma_{\mathbf{0}}, \varepsilon) - 1$, selling after $h_*^{t_1(\sigma_0,\varepsilon)-1}$, and otherwise trading according to $\sigma'_{\mathbf{0}}$. Mimicking the steps in step 4 of the proof of the last lemma, we see that

$$U_{1,T'}(\mathbf{0}, \sigma'_{\mathbf{0}}, p') < \left[k_{\mathbf{0}}(\sigma_{\mathbf{0}}, \varepsilon, \delta)\pi T' + \frac{1}{g(-\bar{y})} \right] \bar{x}.$$

But then $T' < \mathbf{T}(\sigma_{\mathbf{0}})$.

Step 3: The collection $\{B(\delta, \sigma_{\mathbf{0}})\}_{\sigma_{\mathbf{0}} \in \Sigma_{\mathbf{0}}^{nm}}$ is an open covering of the compact set $\Sigma_{\mathbf{0}}^{nm}$. So there exists a finite subcovering $\{B(\delta, \sigma_{\mathbf{0}}^{m})\}_{m=1}^{M}$. Let $\mathbf{T}^*(\mu, \pi, g, c) = \max_m \mathbf{T}(\sigma_{\mathbf{0}}^{m})$. For $T \geqslant \mathbf{T}^*$ there does not exist any $\sigma_{\mathbf{0}} \in \Sigma_{\mathbf{0}}^{nm}$ and $\sigma_{\mathbf{1}} \in \Sigma_{\mathbf{1}}^{nm}$, which is part of an equilibrium of $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$. Further, $\mathbf{T}^*$ does not depend on $\sigma_{\mathbf{N}}$ as none of the $\mathbf{T}(\sigma_{\mathbf{0}}^{m})$ associated with the subcovering do. This completes the proof of the lemma. □

To complete the proof of the theorem, suppose $T \geqslant \mathbf{T}^*(\mu, \pi, g, c)$ and suppose that there exists an equilibrium $\{\sigma_{\mathbf{0}}, \sigma_{\mathbf{1}}, p, Q\}$ of the game $\Gamma(\mu, \pi, g, c, \sigma_{\mathbf{N}}, T)$ such that given p, neither $\sigma_{\mathbf{0}}$ nor $\sigma_{\mathbf{1}}$ involve manipulation, *on the path of play*.[14] Then, it is straightforward to check that the bound on date 1 expected payoffs obtained in Lemma 2 also applies for each $\theta \in \{0,1\}$, so that the profitable deviation of Lemma 3

---

[14] That is σ_θ is non-manipulative for all x^t, h^t such that $q(x^t, h^t | \sigma_\theta) > 0$.

also applies. Thus, for $T \geqslant \mathbf{T}^*(\mu, \pi, g, c)$ we must indeed have manipulation on the path of play.

References

Allen, F., Gale, D., 1992. Stock price manipulation. The Review of Financial Studies 5, 503–529.

Allen, F., Gorton, G., 1992. Stock price manipulation, market microstructure and asymmetric information. European Economic Review 36, 624–630.

Bernhardt, D., Davies, B., Spicer, J., 2002. Long-term information, short-lived derivative securities. Mimeo, Queen's University.

Brunnermeier, M., 2000. Buy on rumors—sell on news: a manipulative trading strategy. Mimeo, Princeton University.

Chakraborty, A., Yılmaz, B., 2000. A note on manipulation. Mimeo, University of Pennsylvania.

Chakraborty, A., Yılmaz, B., 2004. Informed manipulation. Journal of Economic Theory, forthcoming.

Fishman, M.J., Hagerty, K., 1995. The mandatory disclosure of trades and market liquidity. Review of Financial Studies 8, 637–676.

Huddart, S., Hughes, J., Levine, C., 2001. Public disclosure and dissimulation of insider trades. Econometrica 69, 665–681.

Jarrow, R., 1992. Market manipulation, bubbles, corners, and short squeezes. Journal of Financial and Quantitative Analysis 27, 311–336.

John, K., Narayanan, R., 1997. Market manipulation and the role of insider trading regulations. Journal of Business 70, 217–247.

Kumar, P., Seppi, D.J., 1992. Futures manipulation with cash settlement. Journal of Finance 47, 1485–1502.

Kyle, A.S., 1984. A theory of futures market manipulation. In: Anderson, R.W. (Ed.), The Industrial Organization of Futures Markets, Lexington Books, Lexington, MA.

Kyle, A.S., 1985. Continuous auctions and insider trading. Econometrica 53, 1315–1335.

Moore, J.C., 1999. Mathematical Methods for Economic Theory. Springer, Berlin.