1. <u>Signal Processing Fundamentals</u>

1.1 Continuous and discrete signals

Fundamental to the study of radio science and techniques for space exploration is the concept of a signal and how the signal is represented and processed. In this course we want to use the word "signal" in the broadest possible sense.

A <u>signal</u> is the variation of a physically measurable quantity. Variations of the quantity are manifestations of a particular process under study and have this process as their cause.

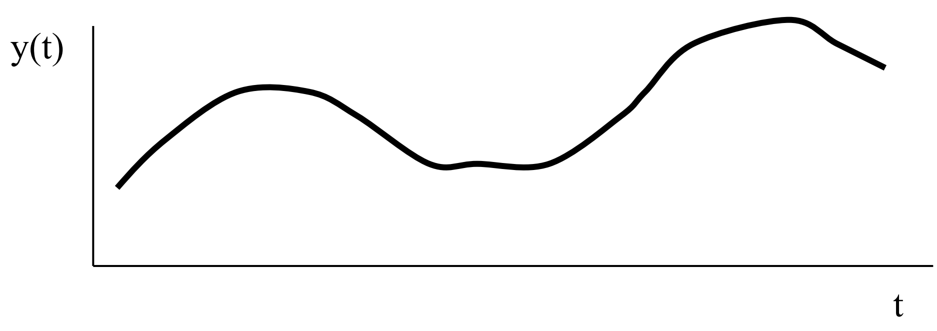
<u>Noise</u>, in contrast, is a variation of the physically measurable quantity. However, variations of the quantity are not a manifestation of a particular process under study and have a frequently unknown process as their cause.

Examples of signals:

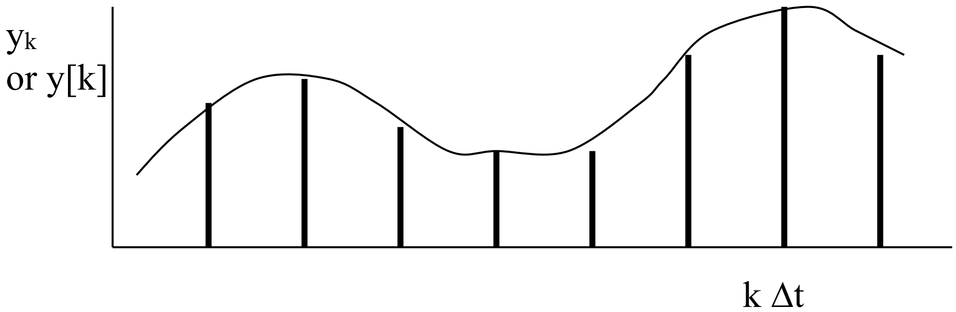
- The voltage fluctutions at the focus of a radio antenna tracking the Cassini spacecraft.
- The current across a 1 Ohm resistor.
- The beam pattern of a radar antenna
- The ground displacements in continental drift

In electrical engineering a signal is often a function of time and/or space and can be functionally represented.

Example: a continuous time signal:



If we assume that the function y(t) has been sampled at regular intervals, Δt, then the set of samples is called a time series. It consists of N known values, y_k, k=1,2,3,.., N



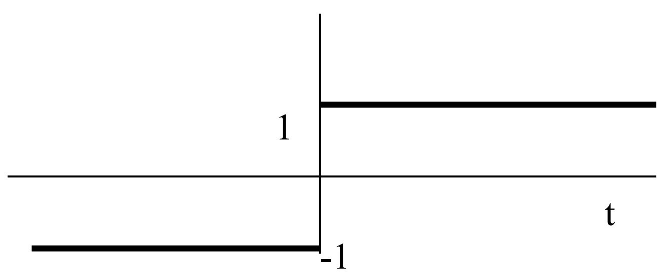
In the following we want to use parenteses for continuous functions and subscripts or brackets for discrete functions. For instance:

$y(t)$: continuous function
y_k : discrete function
$y[k]$; discrete function

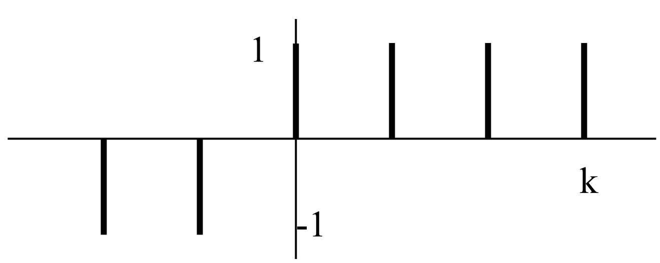
Some very fundamental continuous and discrete functions are:

<u>Sign function</u>

$$\mathrm{sgn}(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t < 0 \end{cases}$$

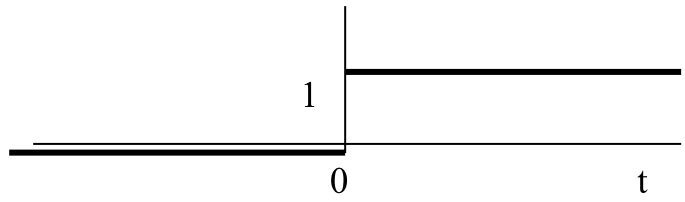


$$\mathrm{sgn}[k] = \begin{cases} 1, & k \geq 0 \\ -1, & k < 0 \end{cases}$$

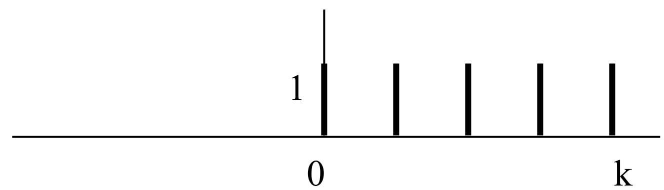


Step function

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

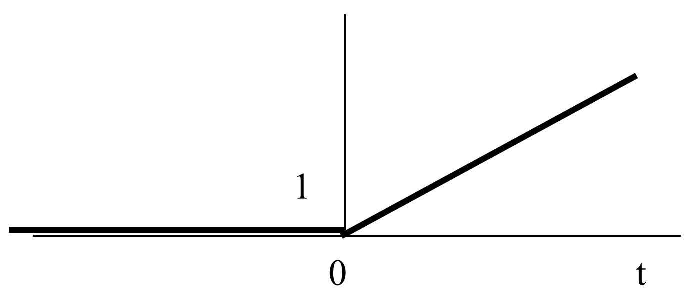


$$u[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

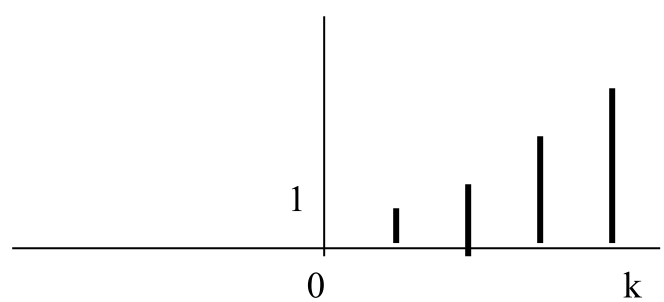


Ramp function

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

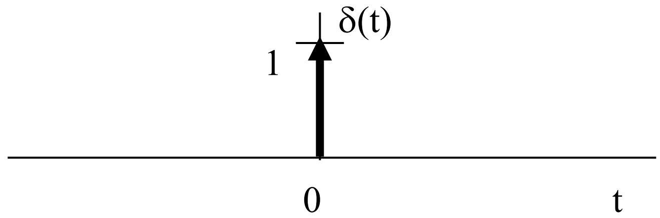


$$r[k] = \begin{cases} k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$



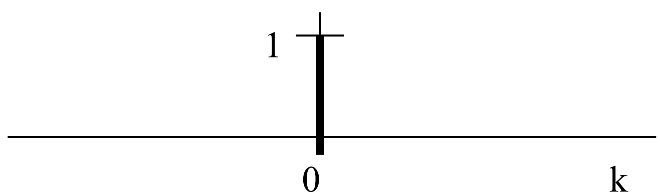
Delta function

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$



$$\int_{-\infty}^{+\infty} \delta(t) = 1$$

$$\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$



Some important relations are:

$u(t) = ½ (1+sgn(t))$
$u[k]=1/2(1+sgn[k])$

$r(t) = t\, u(t)$
$r[k]=k\, u[k]$

$\delta(t) = d/dt\, u(t)$
$\delta[k] =u[k] - u[k-1], \quad k\leq 0$

$$u(t) = \int_{-\infty}^{t} \delta(t)\, dt$$

$$u[k] = \sum_{n=-\infty}^{k} \delta[k]$$

1.2 Fourier Series

The representation of a function over a certain interval by a linear combination of mutually orthogonal functions is called a Fourier series representation of a function. Examples of sets of orthogonal functions are:

Trigonometric functions:	$\sin n\omega_0 t$, $\cos n\omega_0 t$
Exponential functions:	$\exp(i\, n\omega_0 t)$
Legrendre polynomials:	$P_n(t)=1/(2^n\, n!)\ d^n/dt^n\ (t^2-1)^n$
Bessel functions:	$J_n(\beta) = 1/(2\pi) \int_{-\pi}^{+\pi} \exp(i\, \beta \sin x - nx)\ dx$

One particular Fourier series we will use in this course is the exponential Fourier series. It uses the functions { $\exp(i\, n\omega_0 t)$ } $n=0, \pm1, \pm2, \pm3,\ldots$ which are orthogonal over the interval (t_0, t_0+T).

$$y(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega_0 t}$$

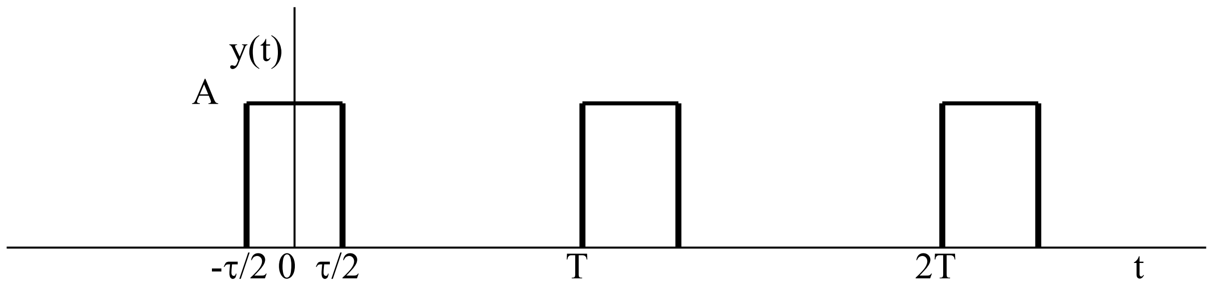
$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} y(t) e^{-in\omega_0 t} dt$$

The magnitude and phase of the n^{th} harmonic are

$$|c_n| = \sqrt{\mathrm{Re}^2[c_n] + \mathrm{Im}^2[c_n]}$$

$$\angle c_n = \tan^{-1}\left(\frac{\mathrm{Im}[c_n]}{\mathrm{Re}[c_n]}\right)$$

Example: Rectangular pulse train with period T



We can now compute the coefficients, c_n, by writing the function y(t) in the form;

$$y(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} = 2\pi\nu_0 \quad \text{fundamental frequency}$$

The coefficients are then given as:

$$c_n = \frac{1}{T}\int_{-\tau/2}^{+\tau/2} A e^{-in\omega_0 t} dt$$

$$c_n = \frac{A}{T}\int_{-\tau/2}^{+\tau/2} e^{-in\omega_0 t} dt$$

$$c_n = \frac{A}{T}\frac{1}{-in\omega_0} e^{-in\omega_0 t}\Big|_{-\tau/2}^{+\tau/2}$$

$$c_n = \frac{A}{T}\frac{1}{n\omega_0}(\sin n\omega_0 t + i\cos n\omega_0 t\ \Big|_{-\tau/2}^{+\tau/2}$$

$$c_n = \frac{A}{n\pi}(\sin\frac{n\pi\tau}{T})$$

$$c_0 = \lim_{n\to 0}\frac{A}{n\pi}(\sin\frac{n\pi\tau}{T})$$

$$c_0 = \frac{A\tau}{T}$$

And the magnitude and phase part of the coefficients are:

$$\left|c_n\right| = \left|\frac{A}{n\pi}(\sin\frac{n\pi\tau}{T})\right|$$

$$\angle c_n = \tan^{-1}(\frac{0}{c_n})$$

Let us now look at a specific example, for instance a pulsed radar signal for range and range rate determinations, and assume that:

T = 4ms
τ = 1ms.

We get discrete values for c_n with an envelope that peaks at n=0 and has zero-crossings at

$n\pi\tau/T = \pi$
$n\tau/T = 1$
$n\nu_0\tau = 1$ with $T=2\pi/\omega_0 = 1/\nu_0$

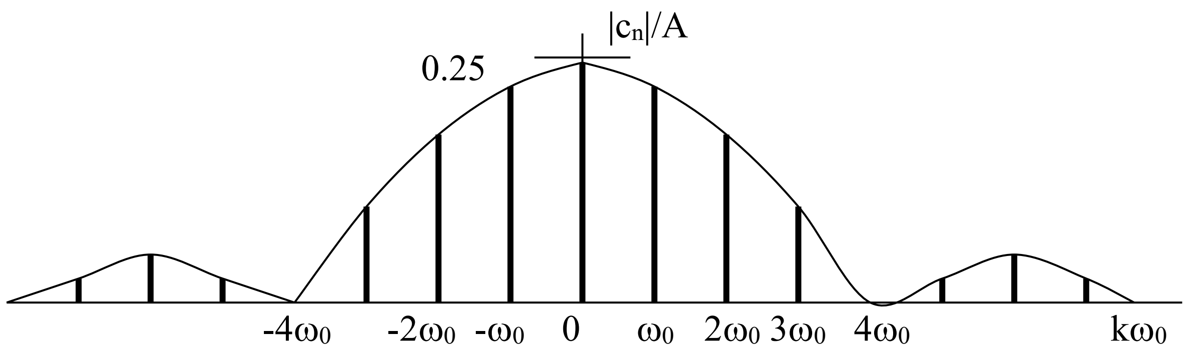
$n\nu_0 = 1/\tau$
= 1 kHz

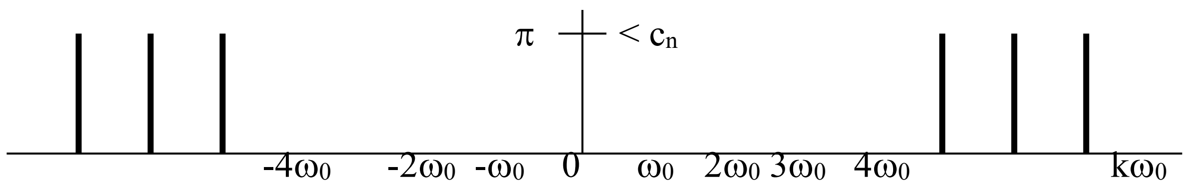
$\nu_0 = 1/T$
= 250 Hz

n = 4

Plugging in several values of n, we get:

n	0	±1	±2	±3	±4	±5	±6	±7	±8	±9
$\|c_n\|/A$	0.250	0.225	0.159	0.075	0	0.045	0.053	0.032	0	0.025
$< c_n$	0	0	0	0	-	π	π	π	-	0





Note: The phases at $-3\omega_0$ to $3\omega_0$ are zero, but the phases at $-4\omega_0$ and $4\omega_0$ are not defined. There is a difference!

Plots of the magnitude and phase of c_n are called <u>magnitude and phase spectra.</u>

Now we are in a good shape to do some Fourier series mental gymnastics.

Here are some questions:

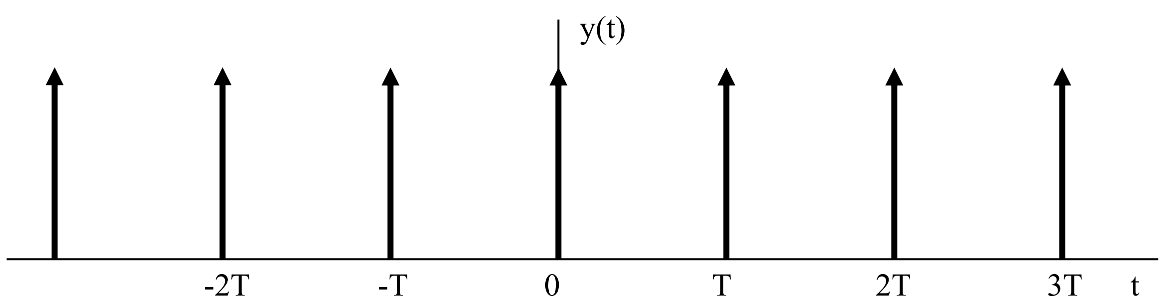
How do the magnitude and phase spectra change when we

- Increase A to 2A
- Increase T to 2T but leave $\tau/T = 1/4$
- Increase τ/T from ¼ to 1/2
- Decrease τ/T from ¼ to 1/8

One of these gymnastic exercises is of particular interest:

Example: Unit impulse train

$\tau \to 0$, $A\tau = 1$, T=const.



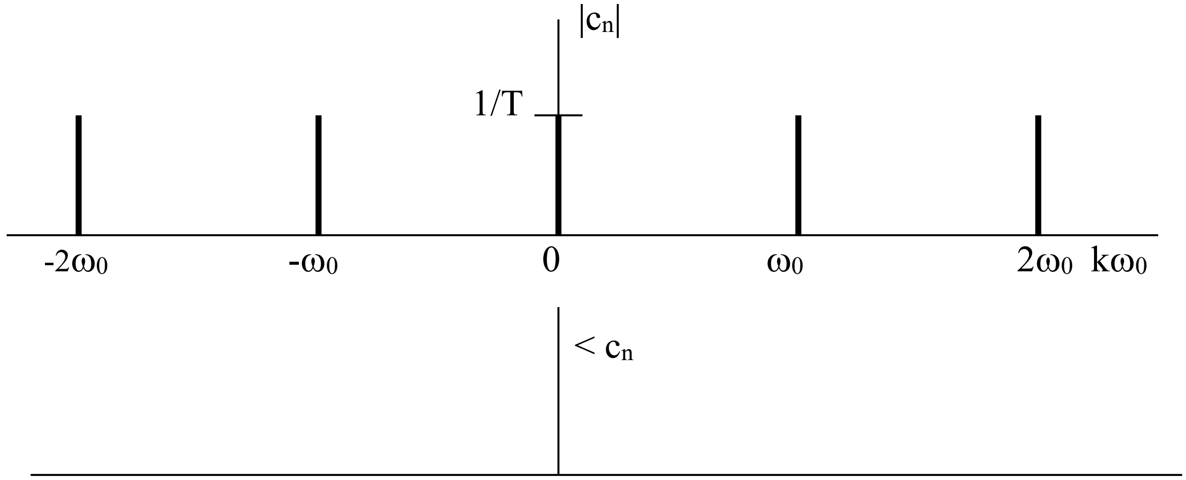
$$y(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$c_n = \lim_{\tau \to 0, A\tau = 1} \frac{A}{n\pi} (\sin \frac{n\pi\tau}{T})$$

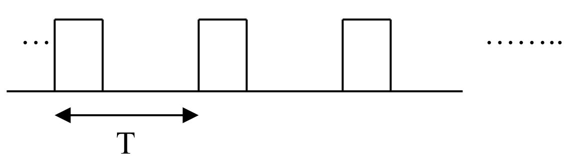
$$c_n = \lim_{\tau \to 0, A\tau = 1} \frac{A}{n\pi} (\frac{n\pi\tau}{T})$$

$$c_n = \frac{1}{T}$$

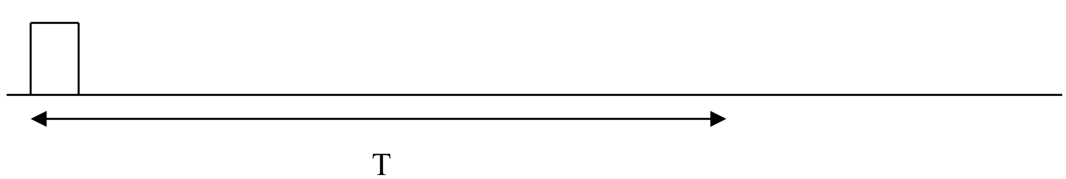
$$y(t) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$



Note:
For a periodic signal, the Fourier series is an accurate expression for **all time** even though the integration for the computation of the coefficients is carried out over only one period.



For a non-periodic signal, the fourier series is an accurate expression only **over the time interval assumed to be one period.**

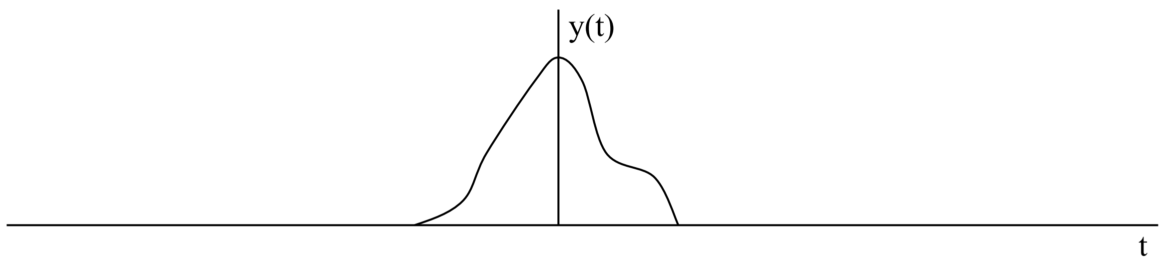


1.3 Fourier transform

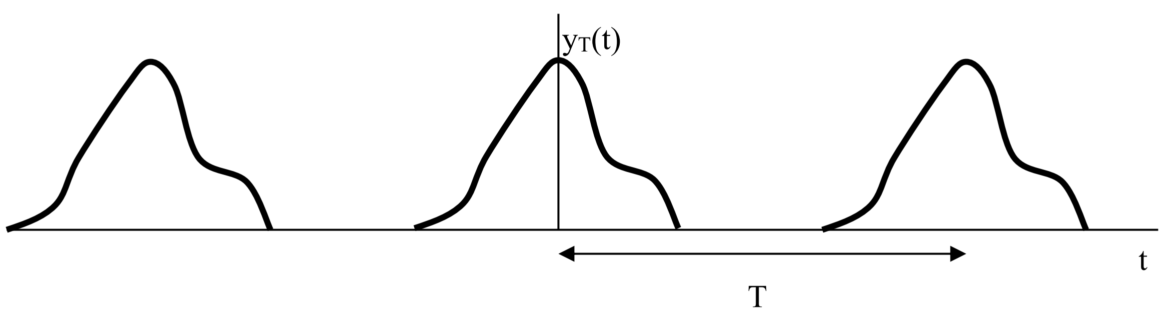
The exponential Fourier series (FS) is an extremely useful technique for the representation of periodic signals. They are also used for non-periodic signals for specific time intervals, or more generally, variable intervals, outside which the accuracy of the representation is unimportant.

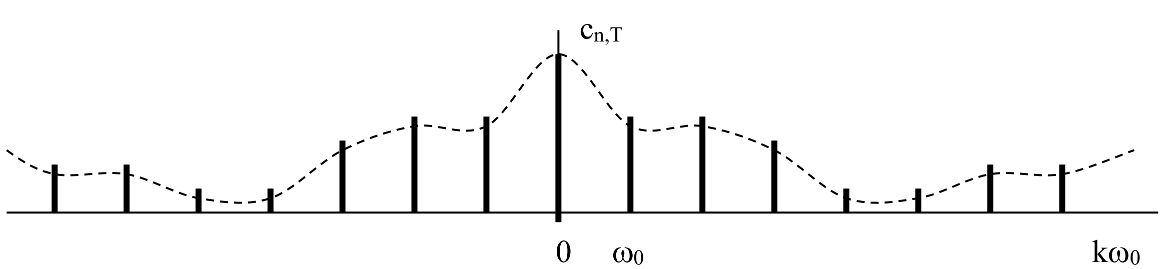
The Fourier transform (FT) is used for the representation of a non-periodic signal that is valid for all time, or more generally, for the whole range of the variable.

The FT is obtained from the exponential FS via a limiting argument.
Let us assume that we have an arbitrary function given below.



This is a non-periodic signal. To derive the FT from the FS we want to first consider the periodic version of this signal, $y_T(t)$, with period T, and its (assumed) spectrum, $c_{n,T}$ as sketched below.



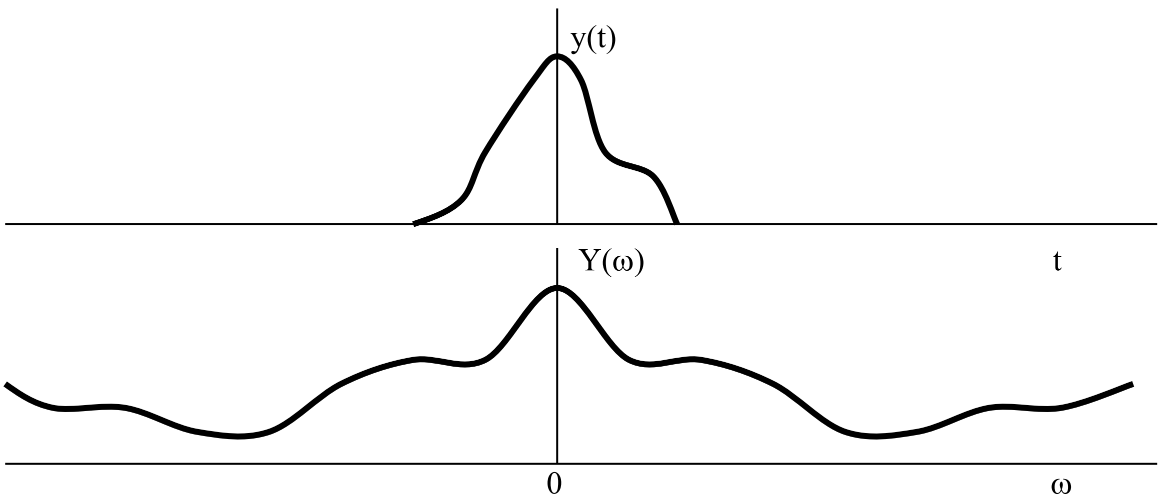


Hopefully we are sufficiently fit through our previous Fourier gymnastics that we can now answer the following question:

What happens with the spectrum if we increase the period T but leave the shape of the pulses unchanged?

If the period T is increased, the fundamental frequency, ω_0, decreases, the spectrum becomes denser, but the shape of the envelope of the spectrum remains (save for a scaling factor) unchanged.

In the limit:
$T \rightarrow \infty$,
$f_T(t) \rightarrow f(t)$
$c_{n,T} \rightarrow Y(\omega)$
discrete spectrum → continuous spectrum



$$y(t) = \frac{1}{2\pi} \bullet \int_{-\infty}^{+\infty} Y(\omega) e^{i\omega t} d\omega$$

$$Y(\omega) = \int_{-\infty}^{+\infty} y(t) e^{-i\omega t} dt$$

y(t) and Y(ω) are called: <u>FT pair</u> : *y(t)↔Y(ω)*

Alternative expressions, with $\omega = 2\pi\nu$, are:

$$y(t) = \int_{-\infty}^{+\infty} Y(\nu) e^{i2\pi\nu t} d\omega$$

(should be dν)

$$Y(\nu) = \int_{-\infty}^{+\infty} y(t) e^{-i2\pi\nu t} dt$$

In order for the FT to exist, we must have $Y(\omega) < \infty$.
How can we find out?

1) Evaluate the integral
2) Consider Dirichlet conditions

Dirichlet Conditions:

If
1)

$$\int_{-\infty}^{+\infty} |y(t)| \, dt < \infty$$

or

$$\int_{-\infty}^{+\infty} |y(t)|^2 dt < \infty$$

and

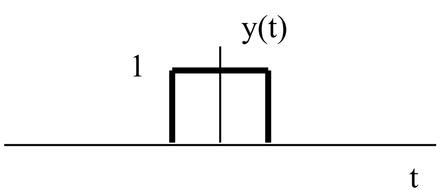
2) y(t) has a finite number of maxima and minima in any finite interval, and
3) y(t) has a finite number of discontinuities in any finite interval

<u>then FT exists</u>.

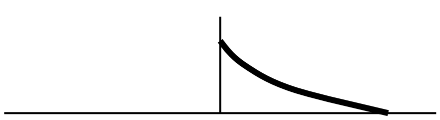
Note: If Dirichlet conditions are not fulfilled, then FT could perhaps still exist.

Examples

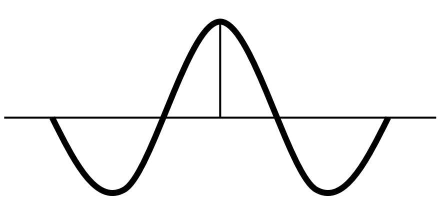
Here are four examples that do and do not fulfill the Dirichlet conditions:



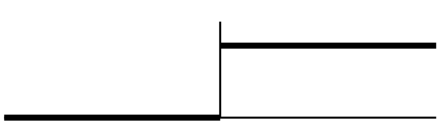
$$y(t)=\Pi(t) = \begin{cases} 1, & |t| \le 1/2 \\ 0, & |t| > 1/2 \end{cases}$$



$y(t)=u(t)e^{-at}$, $a>0$



$y(t)=\cos \omega_0 t$

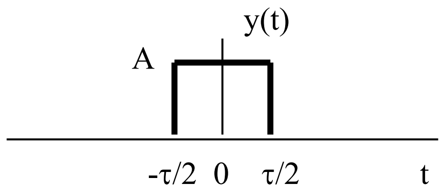


$y(t)=u(t)$

Which do and which do not fulfill the Dirichlet conditions? Here is the answer: The first two do and the last two do not.

Now, let us compute the FTs.

1st example: FT of the gate function



$$y(t)=A\Pi(t/\tau) = \begin{cases} A, & |t| \le \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

$$Y(\omega) = A\int_{-\tau/2}^{+\tau/2} e^{-i\omega t}\,dt$$

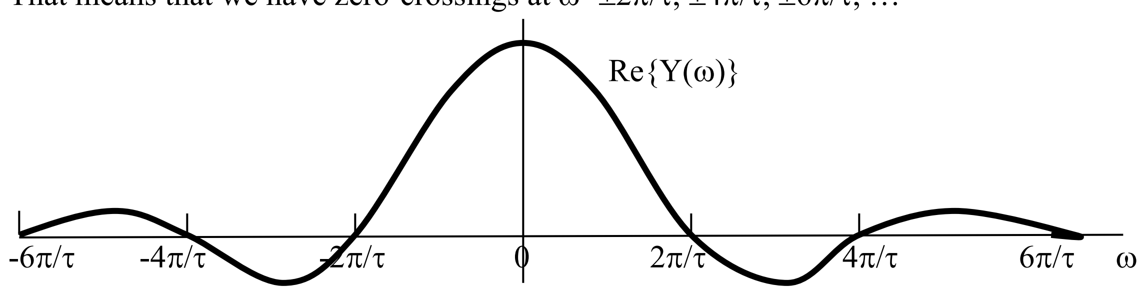
$$Y(\omega) = -\frac{A}{i\omega} e^{-i\omega t} \Big|_{-\tau/2}^{+\tau/2}$$

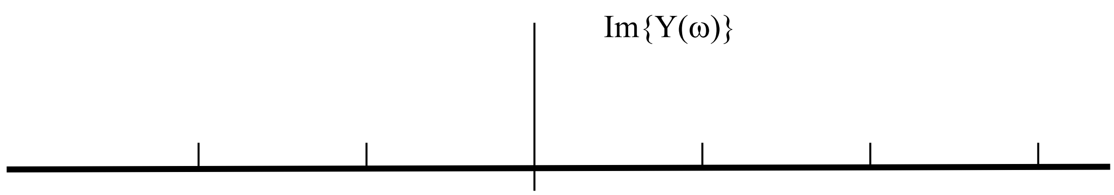
$$Y(\omega) = \frac{A}{i\omega}\left(e^{i\omega\frac{\tau}{2}} - e^{-i\omega\frac{\tau}{2}}\right)$$

$$Y(\omega) = \frac{2A}{\omega}\sin\frac{\omega\tau}{2}$$

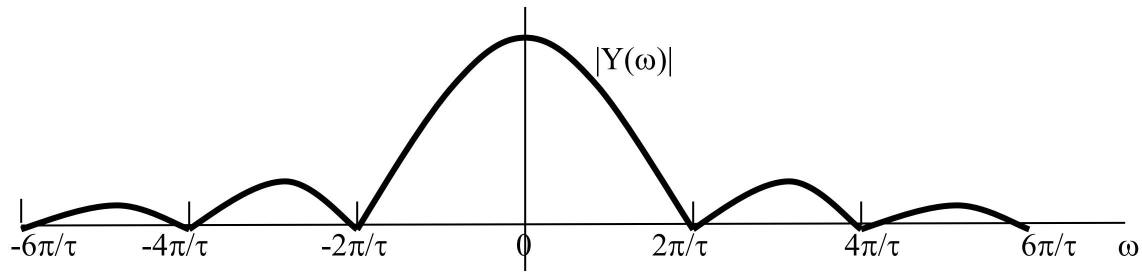
$$Y(\omega) = A\tau\frac{\sin\omega\frac{\tau}{2}}{\omega\frac{\tau}{2}}$$

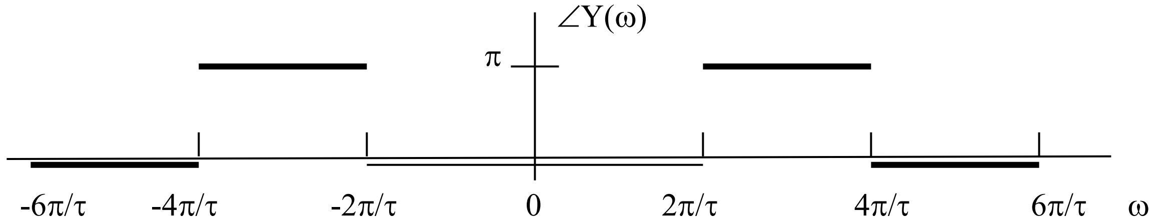
$Y(\omega)$ is a real function and has zero-crossings at $\omega\tau/2 = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$
That means that we have zero-crossings at $\omega = \pm 2\pi/\tau, \pm 4\pi/\tau, \pm 6\pi/\tau, \ldots$





We can also plot the magnitude and phase spectra:



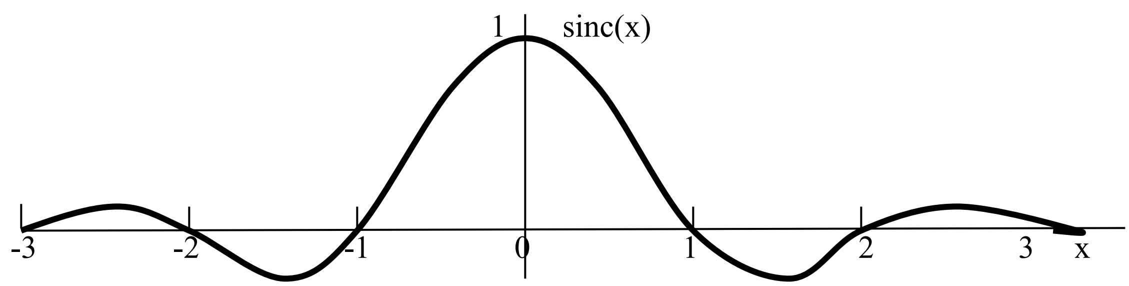


The function of the form $\frac{\sin x}{x}$ plays an important role in signal theory.

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

Sinc function:

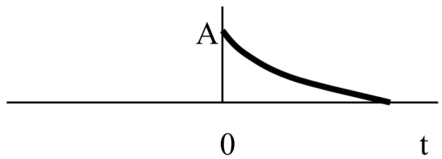
$$\Pi(\frac{t}{\tau}) \Leftrightarrow \tau \operatorname{sinc}(\frac{\omega\tau}{2\pi})$$



For the gate function: $\Pi(\frac{t}{\tau})$ we can now write:

$$\Pi(\frac{t}{\tau}) \Leftrightarrow \tau \operatorname{sinc}(\frac{\omega\tau}{2\pi})$$

2nd example: FT of the exponentially decaying function



y(t)=Au(t)e^{-at}, a>0

$$Y(\omega) = A\int_0^{+\infty} e^{-(a+i\omega)t}\,dt$$

$$Y(\omega) = -\frac{A}{a+i\omega} e^{-(a+i\omega)t}\Big|_0^{+\infty}$$

$$Y(\omega) = \frac{A}{a+i\omega}$$

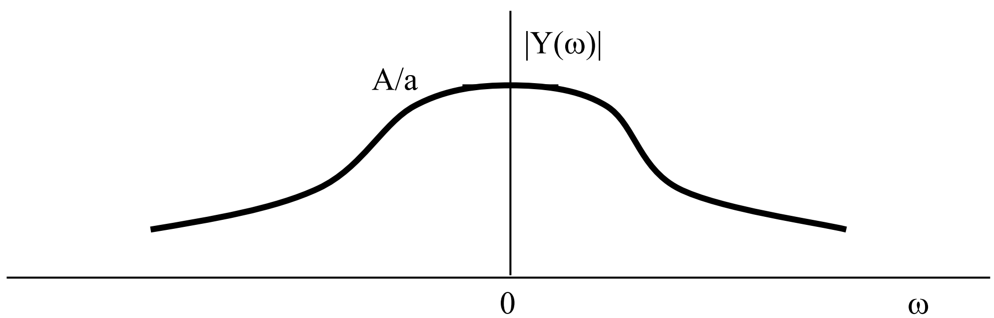
$$Y(\omega) = \frac{A}{a^2+\omega^2}(a-i\omega)$$

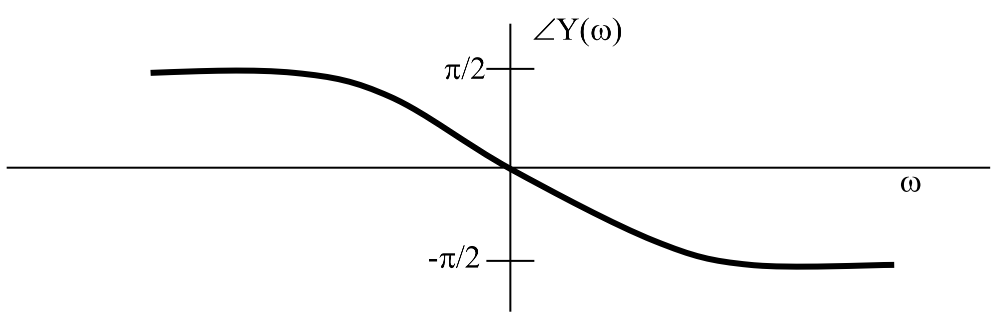
$$|Y(\omega)| = \frac{A}{a^2+\omega^2}(a^2+\omega^2)^{1/2}$$

$$|Y(\omega)| = \frac{A}{(a^2+\omega^2)^{1/2}}$$

$$\angle Y(\omega) = \tan^{-1}\frac{-\omega}{a}$$

$$\angle Y(\omega) = -\tan^{-1}\frac{\omega}{a}$$





The two other functions do not fulfill the Dirichlet conditions, however, their FT exists anyway. For the computation of the FT of these two other functions we need the δ-function which is also called the unit impulse function:

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

Properties of the δ-function

1) Shifting property or sampling property:

$$\int_{-\infty}^{+\infty} f(t)\delta(t)\,dt = f(0)$$

$$\int_{-\infty}^{+\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

$$\int_{-\infty}^{+\infty} f(t-t_1)\delta(t-t_2)\,dt = f(t_2-t_1)$$

2) Scaling property:

$$\int_{-\infty}^{+\infty} f(t)\delta(at)\,dt = \frac{1}{|a|} f(0)$$

3) derivative property:

$$\int_{-\infty}^{+\infty} f(t)\delta^{(n)}(t-t_0)\,dt = (-1)^n f^{(n)}(t)\Big|_{t=t_0}$$

Now we can compute the FT of the δ-function:

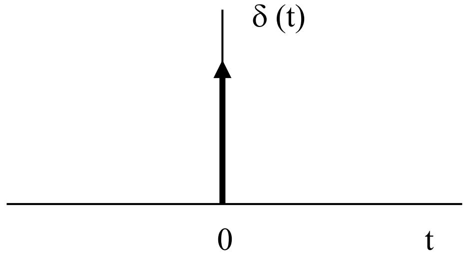
$$\mathrm{FT}\{\delta(\mathrm{t})\} = \int_{-\infty}^{+\infty} \delta(t)e^{-i\omega t}\,dt$$

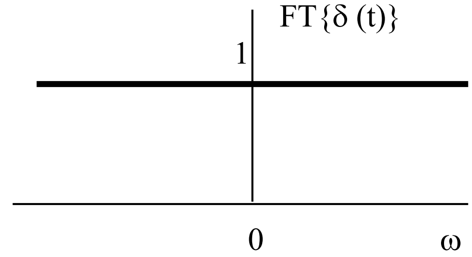
$$\mathrm{FT}\{\delta(\mathrm{t})\} = 1$$

And since FT is unique, δ(t) is the inverse FT of 1

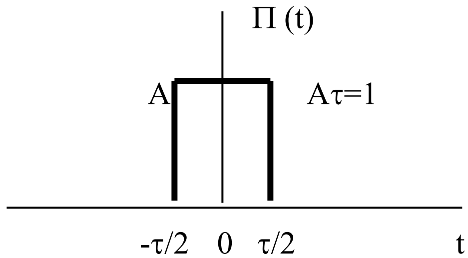
$$\delta(\mathrm{t}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 1 e^{i\omega t}\,d\omega$$

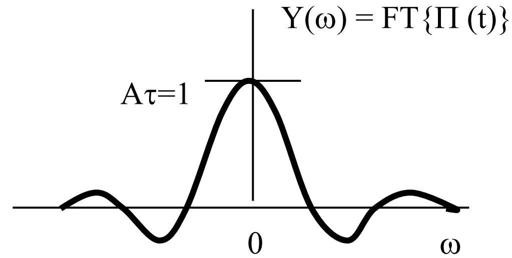
$$\delta(\mathrm{t}) \leftrightarrow 1$$





Note: This result can also be obtained through a limiting argument;





FT of a constant:

$$\mathrm{FT}\{1\} = \int_{-\infty}^{+\infty} 1 e^{-i\omega t} dt$$

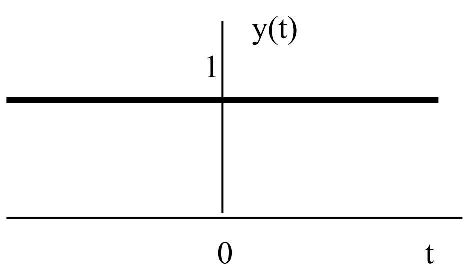
this is not absolutely integrable. δ-function is needed. Also we have to do a change of variables to be able to use a previous equation.
t = - x => dt = -dx, interval boundaries need a sign change too.

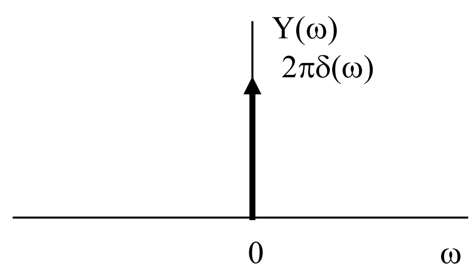
$$\mathrm{FT}\{1\} = \int_{-(-\infty)}^{-(+\infty)} 1 e^{-i\omega(-x)}(-dx)$$

$$\mathrm{FT}\{1\} = \int_{-\infty}^{+\infty} 1 e^{i\omega x} dx$$

$$\mathrm{FT}\{1\} = 2\pi\delta(\omega)$$

$$1 \Leftrightarrow 2\pi\delta(\omega)$$





Now it gets really exciting since we can now compute the FTs of the other two example functions.

3rd example: FT of the cos function

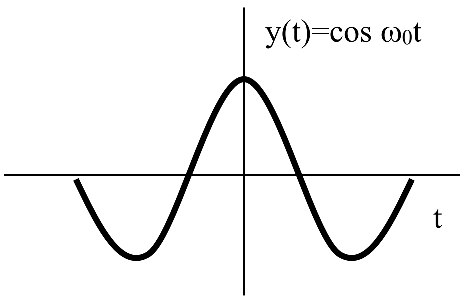
$$\mathrm{FT}\{\cos\omega_0 t\} = \frac{1}{2}\mathrm{FT}\{e^{i\omega_0 t} + e^{-i\omega_0 t}\}$$

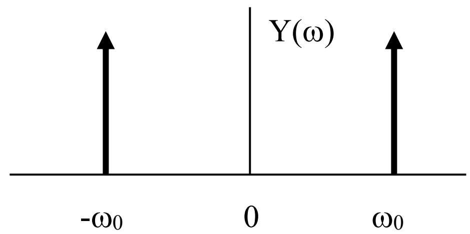
$$\mathrm{FT}\{\cos\omega_0 t\} = \frac{1}{2}[\int_{-\infty}^{+\infty} e^{-i(\omega-\omega_0)t}dt + e^{-i(\omega+\omega_0)t}dt]$$

And again with change of variables: t = -x

$$\mathrm{FT}\{\cos\omega_0 t\} = \frac{1}{2}[\int_{-\infty}^{+\infty} e^{i(\omega-\omega_0)x}dx + e^{i(\omega+\omega_0)x}dx]$$

$$\mathrm{FT}\{\cos\omega_0 t\} = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$





Note:

The FT was originally defined for non-periodic signals. Now we find FT's also for certain periodic signals through the use of the δ-function. In fact we can find the FT for any general, periodic function! The procedure is to write down the complex exponential FS for a function f(t) and then take the FT of the series on a term to term basis.

$$\mathrm{FT}\{\mathrm{y(t)}\} = \mathrm{FT}\{\sum_{n=-\infty}^{+\infty} c_n e^{in\omega_0 t}\}$$

$$\mathrm{FT}\{\mathrm{y(t)}\} = \sum_{n=-\infty}^{+\infty} c_n \mathrm{FT}\{e^{in\omega_0 t}\}$$

$$\mathrm{FT}\{\mathrm{y(t)}\} = 2\pi\sum_{n=-\infty}^{+\infty} c_n \delta(\omega - n\omega_0)$$

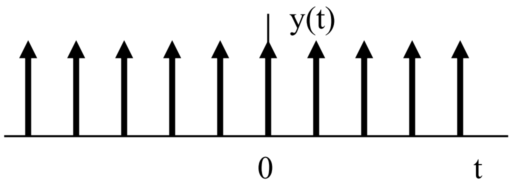
Example: Train of unit impulses

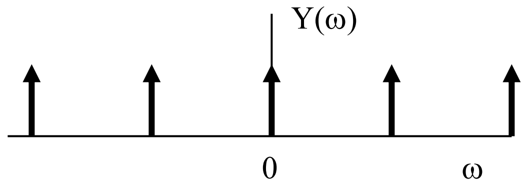
$$\mathrm{y(t)} = \sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

$$\mathrm{y(t)} = \frac{1}{\mathrm{T}}\sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$ (see p.9)

$$Y(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_0)$$

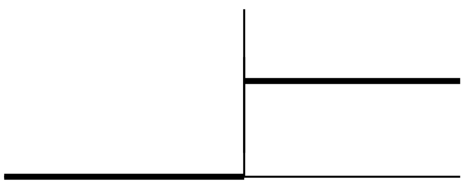
$$Y(\omega)=\omega_0\sum_{n=-\infty}^{+\infty}\delta(\omega-n\omega_0)$$



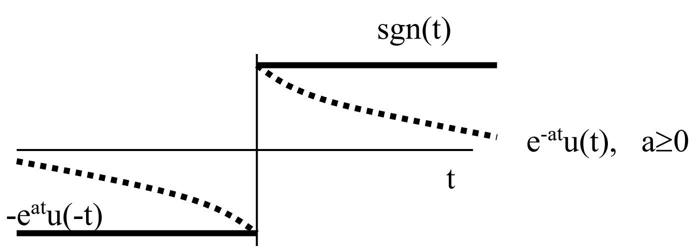


4^{th} example: FT of the step function

$$u(t)=\frac{1}{2}+\frac{1}{2}\operatorname{sgn}(t)$$



First we compute the FT of the sign function through a limiting argument.



$$\operatorname{sgn}(t)=\lim_{a\to 0}[e^{-at}u(t)-e^{at}u(-t)]$$

$$\text{FT}\{\operatorname{sgn}(t)\}=\lim_{a\to 0}[\int_0^{\infty}e^{-at}e^{-i\omega t}dt-\int_{-\infty}^{0}e^{at}e^{-i\omega t}dt]$$

$$\text{FT}\{\operatorname{sgn}(t)\}=\lim_{a\to 0}[\int_0^{\infty}e^{-(a+i\omega)t}dt-\int_{-\infty}^{0}e^{+(a-i\omega)t}dt]$$

$$\text{FT}\{\operatorname{sgn}(t)\}=\lim_{a\to 0}[-\frac{1}{a+i\omega}e^{-(a+i\omega)t}\Big|_0^{\infty}-\frac{1}{a-i\omega}e^{(a-i\omega)t}\Big|_{\infty}^{0}]$$

$$\text{FT}\{\text{sgn}(t)\} = \lim_{a \to 0}\left[\frac{1}{a+i\omega} - \frac{1}{a-i\omega}\right]$$

$$\text{FT}\{\text{sgn}(t)\} = \frac{2}{i\omega}$$

$$\text{sgn}(t) \leftrightarrow \frac{2}{i\omega}$$

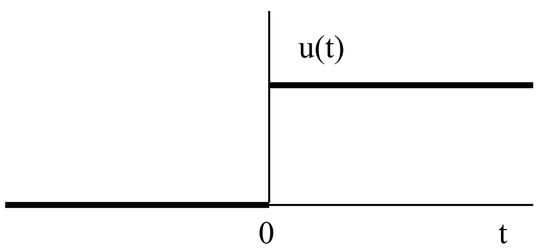
Now we can compute the FT of the step function.

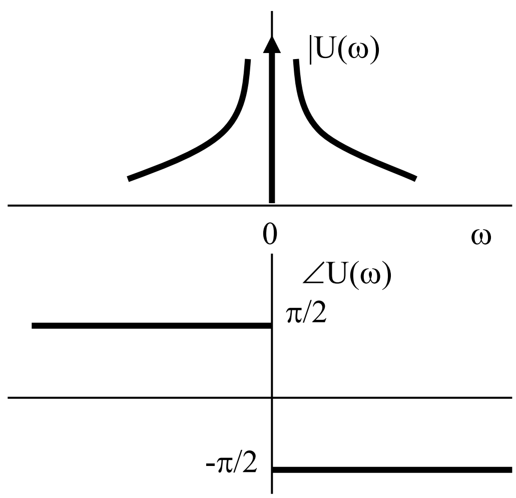
$$\text{FT}\{\frac{1}{2}\} = \frac{1}{2}2\pi\delta(\omega)$$

$$\text{FT}\{\text{u(t)}\} = \pi\delta(\omega) + \frac{1}{i\omega}$$

$$\Re e\text{FT}\{\text{u(t)}\} = \pi\delta(\omega)$$

$$\Im m\text{FT}\{\text{u(t)}\} = -\frac{1}{\omega}$$





1.4 Properties of the Fourier transform

Symmetry property:

If f(t)) $\leftrightarrow$ F(ω) then
F(t)) $\leftrightarrow 2\pi$f(-ω)

Example:
sgn(t) $\leftrightarrow$2/(iω)

$2/(it) \leftrightarrow 2\pi\ \mathrm{sgn}(-\omega)$
$i/(\pi t) \leftrightarrow \mathrm{sgn}(\omega)$

Linearity property:

If $f_1(t) \leftrightarrow F_1(\omega)$ and
$f_2(t) \leftrightarrow F_2(\omega)$, then for arbitrary constants a, b
$af_1(t) + b\, f_2(t) \leftrightarrow aF_1(\omega) + b\, F_2(\omega)$

Note: This is very useful, since it means that you can compute the FT in steps.

Time-shifting property:

If

$$f(t) \leftrightarrow F(\omega)$$
$$f(t-t_0) \leftrightarrow F(\omega)e^{-i\omega t_0}$$ then

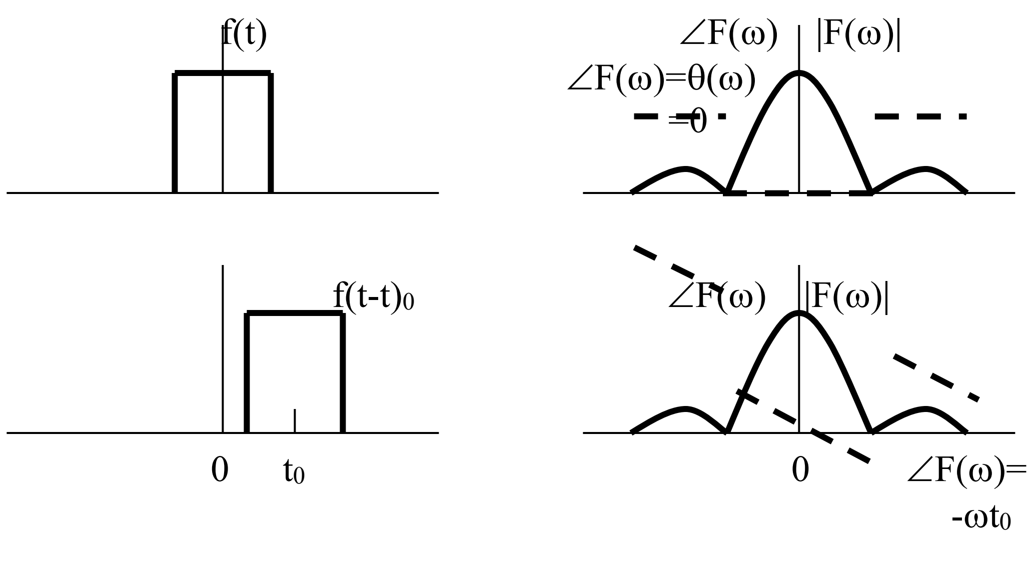
Note: If

$$F(\omega) = |F(\omega)|e^{i\theta(\omega)}$$
$$f(t-t_0) \leftrightarrow |F(\omega)|e^{i(\theta(\omega)-\omega t_0)}$$

A shift of t in the time domain leaves the magnitude spectrum unchanged, but the phase spectrum acquires an additional term - ωt_0, see, for instance:

$$\cos\omega(t-t_0) = \cos(\omega t - \omega t_0)$$

Example:



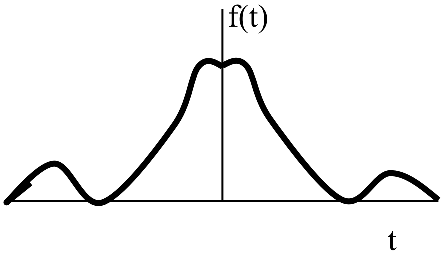
Frequency-shifting property:

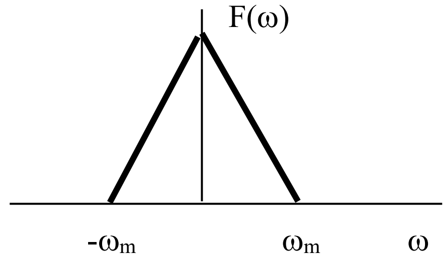
If

$$f(t) \leftrightarrow F(\omega)$$

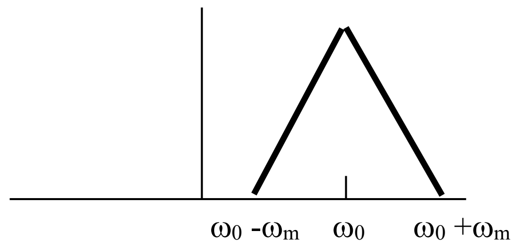
$$f(t)e^{i\omega_0 t} \leftrightarrow F(\omega - \omega_0) \quad \text{then}$$

Example 1:

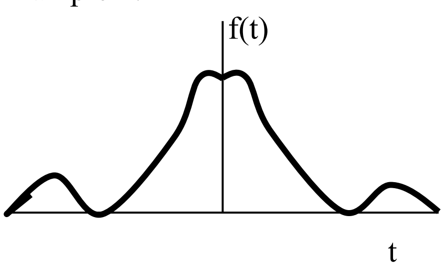


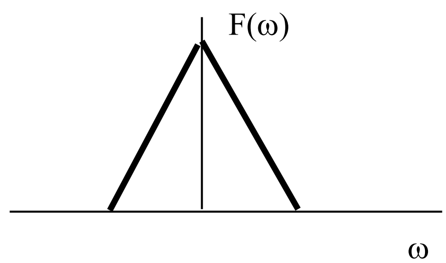


$$f(t)e^{i\omega_0 t}$$

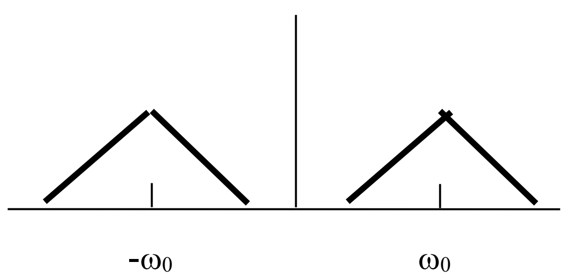


Example 2:

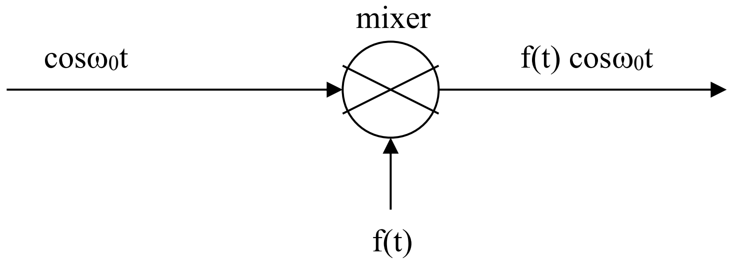




$$f(t)\cos(\omega_0 t)$$



In radio technology you often need to translate a spectrum to a different frequency range, e.g. baseband → IF → RF or RF → IF → baseband. This is achieved through the use of up and down-converters, or mixers.



Time-differentiation property

If

$$f(t) \leftrightarrow F(\omega)$$

then

$$\frac{d}{dt} f(t) \leftrightarrow i\omega F(\omega)$$

$$\frac{d^n}{dt^n} f(t) \leftrightarrow (i\omega)^n F(\omega)$$

Time-integration property

If

$$f(t) \leftrightarrow F(\omega)$$

then

$$\int_{-\infty}^{t} f(x)dx \leftrightarrow \frac{1}{i\omega} F(\omega) + \pi F(0)\delta(\omega)$$

Scaling property

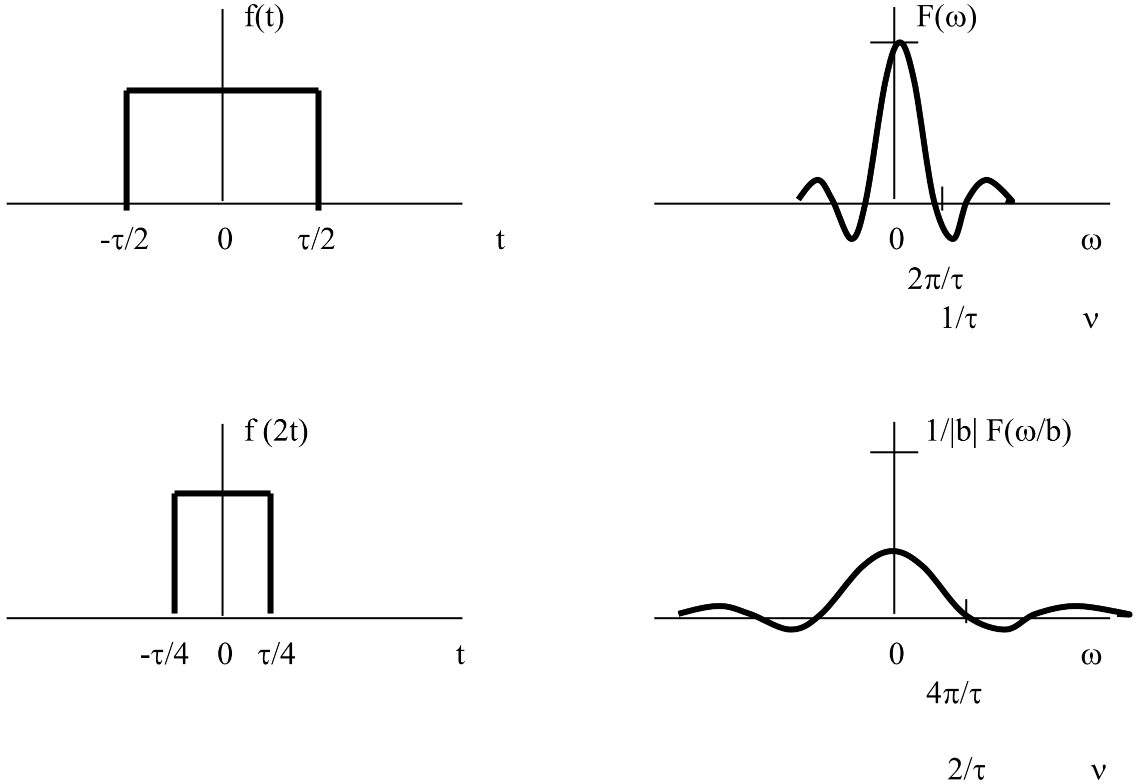
If

$$f(t) \leftrightarrow F(\omega)$$

then for a real constant b

$$f(bt) \leftrightarrow \frac{1}{|b|} F(\frac{\omega}{b})$$

Example:



1.5 The two-dimensional Fourier transform

So far we have used the variables, t and ω or ν in the context of FTs. These variables, time and frequency (radians per second and cycles per second) stand for physical quantities that are one-dimensional. However, in cases which are two-dimensional, an antenna , arrays of antennas, brightness distributions at the sky, pictures on a TV screen, etc., variables that describe two-dimensional quantities need to be used.

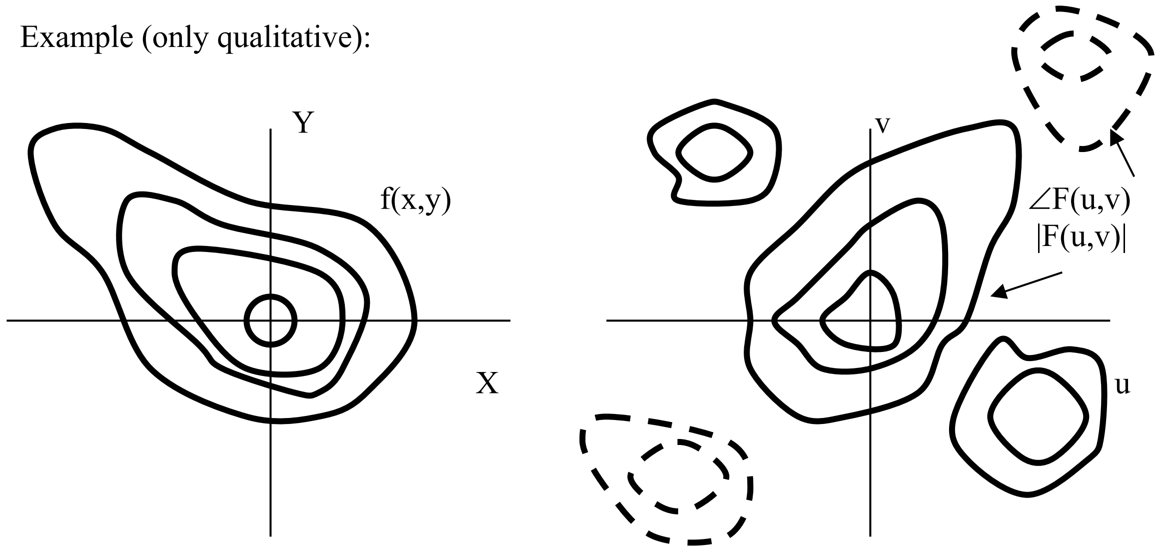
A two-dimensional function f(x,y) has a two-dimensional Fourier transform F(u,v) with

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)e^{i2\pi(ux+vy)}dudv$$

$$F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-i2\pi(ux+vy)}dxdy$$

If x, y are spatial coordinates like angles, then u, v are called <u>spatial frequencies</u>.

Example (only qualitative):



Properties of the two-dimensional Fourier transform

The properties of the two-dimensional FT are very similar to those of the one-dimensional FT. The most important for our purposes are:

Linearity property:

$$af_1(x,y) + bf_2(x,y) \Leftrightarrow aF_1(u,v) + bF_2(u,v)$$

Shifting property:

$$f(x-a, y-b) \Leftrightarrow F(u,v)e^{-i2\pi(au+bv)}$$

Modulation property:

$$f(x,y)e^{i\omega_0 x} \Leftrightarrow F(u-\frac{\omega_0}{2\pi}, v)$$

Scaling property:

$$f(ax,by) \Leftrightarrow \frac{1}{|ab|}F(\frac{u}{a},\frac{v}{b})$$

The extension to more than two dimensions is straight forward.