

1.6 Linear systems, convolution and filtering

We now want to turn our attention to the way in which signal properties are modified by various types of processing.

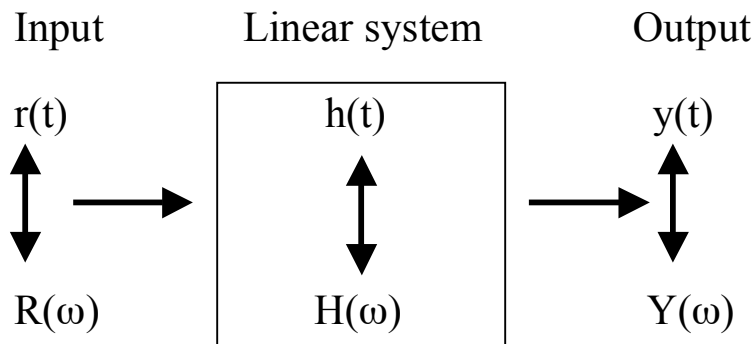
Examples of signal processing devices are:

- amplifiers
- delay lines
- mixers
- filters
- receivers
- transmitters
- antennae
- antenna arrays

Definition of a linear system:

If $\mathbf{r}_1(\mathbf{t})$, $\mathbf{r}_2(\mathbf{t})$ produce system output responses $\mathbf{y}_1(\mathbf{t})$, $\mathbf{y}_2(\mathbf{t})$, respectively, then a system is said to be linear if $a\mathbf{r}_1(\mathbf{t}) + b\mathbf{r}_2(\mathbf{t})$ produces the system output response $a\mathbf{y}_1(\mathbf{t}) + b\mathbf{y}_2(\mathbf{t})$, where a , b are constants.

In general: a linear system modifies the amplitude and phase of each Fourier component of an input signal before it reaches the output.



$h(t)$: impulse response

$H(\omega)$: transfer function

We can write:

$$R(\omega)H(\omega) = Y(\omega)$$

or in magnitude and phase form:

$$|R(\omega)||H(\omega)|e^{i(\theta_r(\omega)+\theta_h(\omega))} = |Y(\omega)|e^{i\theta_y(\omega)}$$

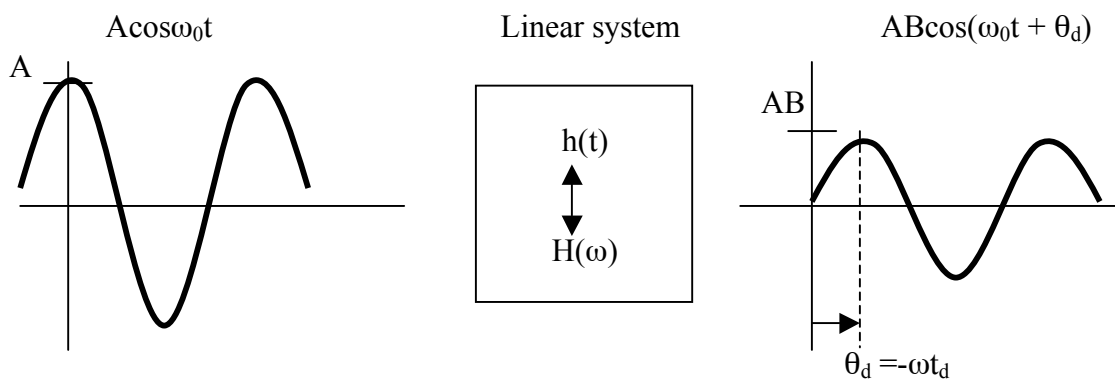
Definition of a time-invariant system:

If $\mathbf{r}(t)$ applied at $t=t_0$, produces a response of a system $\mathbf{y}(t)$, then the system is invariant
 If $\mathbf{r}(t-t_1)$ applied at $t=t_0+t_1$ produces a response $\mathbf{y}(t-t_1)$.

Note:

- Modification characteristics don't change with time.
- Output characteristics contain only those frequencies that are present in the input.
 No new frequencies are generated.

Example of a linear system that is also time-invariant:



The linear system generated:

- amplitude modification by B
- phase shift by θ_d

$$H(\omega) = B e^{-i\theta_d}$$

Example: Delay line

$$H(\omega) = B e^{-i\omega t_d}$$

$$|H(\omega)| = B$$

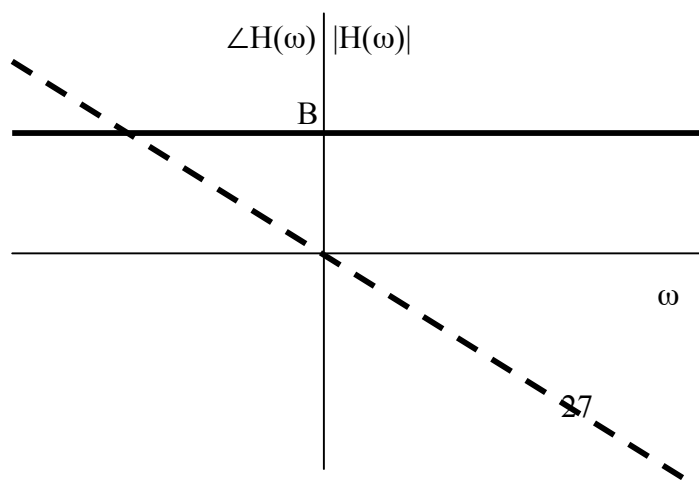
$$\angle H(\omega) = -\omega t_d$$

please note that:

$$\angle H(\omega) = \tan^{-1} \frac{-\sin \omega t_d}{\cos \omega t_d}$$

$$\angle H(\omega) = -\tan^{-1} \tan \omega t_d$$

$$\angle H(\omega) = -\omega t_d$$



$$\theta_h = -\omega t_d$$

Now let us assume that $r(t)$ is just a cos-function;

$$r(t) = A \cos \omega_0 t$$

The its FT is

$$R(\omega) = A\pi[\delta(\omega + \omega_0) + (\omega - \omega_0)]$$

And the FT of the output of the linear system is:

$$R(\omega)H(\omega) = AB\pi[\delta(\omega + \omega_0) + (\omega - \omega_0)]e^{-i\omega t_d}$$

In order to get the output of the linear system in the time domain, we need to compute the inverse FT

$$y(t) = \text{FT}^{-1}\{R(\omega)H(\omega)\}$$

$$y(t) = \frac{AB\pi}{2\pi} \left\{ \int_{-\infty}^{+\infty} \delta(\omega + \omega_0) e^{i\omega(t-t_d)} d\omega + \int_{-\infty}^{+\infty} \delta(\omega - \omega_0) e^{i\omega(t-t_d)} d\omega \right\}$$

$$y(t) = \frac{AB\pi}{2\pi} \left\{ \int_{-\infty}^{+\infty} \delta(\omega + \omega_0) e^{i\omega(t-t_d)} d\omega + \int_{-\infty}^{+\infty} \delta(\omega - \omega_0) e^{i\omega(t-t_d)} d\omega \right\}$$

$$y(t) = AB \frac{1}{2} \left\{ e^{i\omega_0(t-t_d)} + e^{-i\omega_0(t-t_d)} \right\}$$

$$y(t) = AB \cos \omega_0(t - t_0)$$

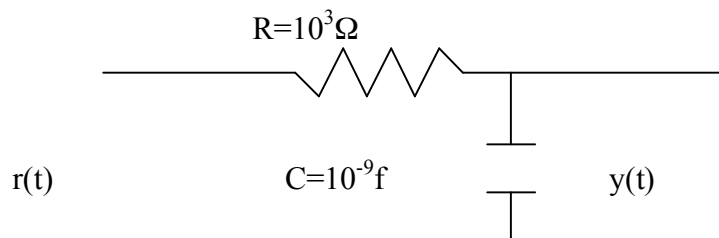
If we had an input signal, $r(t)$ whose spectrum did not consist of only one frequency but Of a range of frequencies, then each of these frequencies would be delayed by t_d . and attenuated (or amplified) by B. The result is a signal delayed by t_d and attenuated by B.

The exact signal shape is reproduced. This is distortion-less transmission.

$$y(t) = Br(t - t_d)$$

In general, however, distortion cannot be avoided and in many cases is not even desired.

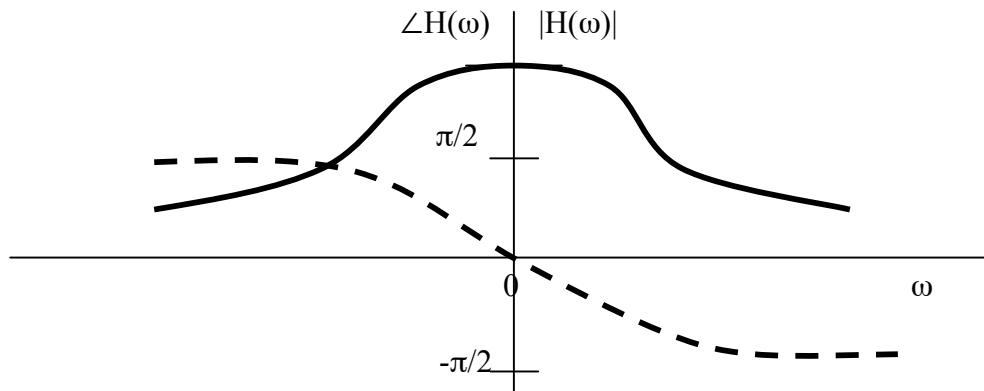
Example: Low-pass filter



The transfer function of a particular low-pass filter is give by

$$H(\omega) = \frac{1}{1 + i\omega RC}$$

Below is a plot of the magnitude and phase spectra



Obviously, a low-pass filter has characteristics of distortion-less transmission only for a narrow bandwidth.

Convolution:

In order to get from $r(t)$ to $y(t)$ in our figure on p. 26, we can use the convolution integral.

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} r(\tau)h(t - \tau)d\tau = \\ &= r(t) * h(t) \end{aligned}$$

Special case: $r(t) = \delta(t)$. Then

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} \delta(\tau)h(t - \tau)d\tau \\ &= h(t) \end{aligned}$$

This is why $h(t)$ is called impulse response.

Proof (for pundits):
We need to show:

$$FT \left\{ \int_{-\infty}^{+\infty} r(\tau)h(t - \tau)d\tau \right\} = R(\omega)H(\omega)$$

There are two ways:

1) We have to change the order of integration and make a change of variables, $\lambda = t - \tau$, so that $d\lambda = dt$, then

$$\begin{aligned} \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} r(\tau)h(t - \tau)d\tau \right\} e^{-i\omega t} dt &= \\ \int_{-\infty}^{+\infty} r(\tau) \left\{ \int_{-\infty}^{+\infty} h(\lambda)e^{-i\omega\lambda} d\lambda \right\} e^{-i\omega\tau} dt &= R(\omega)H(\omega) \end{aligned}$$

2) We can also use the shifting property of the FT

$$\begin{aligned} \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} r(\tau)h(t - \tau)d\tau \right\} e^{-i\omega t} dt &= \\ \int_{-\infty}^{+\infty} r(\tau) \left\{ \int_{-\infty}^{+\infty} h(t - \tau)e^{-i\omega t} dt \right\} d\tau &= \end{aligned}$$

Then with the shifting property of the FT: $f(t - t_0) \leftrightarrow F(\omega)e^{-i\omega t_0}$
we get:

$$\int_{-\infty}^{+\infty} r(\tau)H(\omega)e^{-i\omega\tau} d\tau = R(\omega)H(\omega)$$

Laws of the convolution operation:

* is communicative	$r * h = h * r$
* is distriputive	$r * [h_1 + h_2] = r * h_1 + r * h_2$
* is associative	$r * [h_1 * h_2] = [r * h_1] * h_2$

→ just like multiplication!

Theorems:

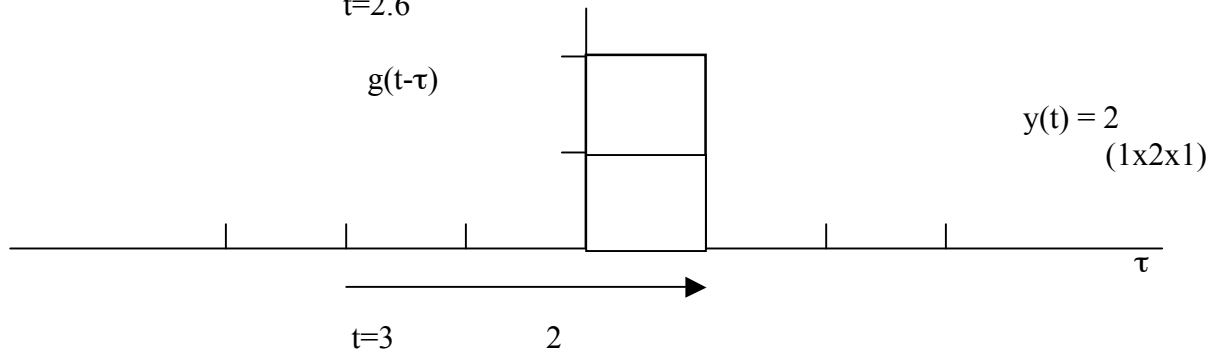
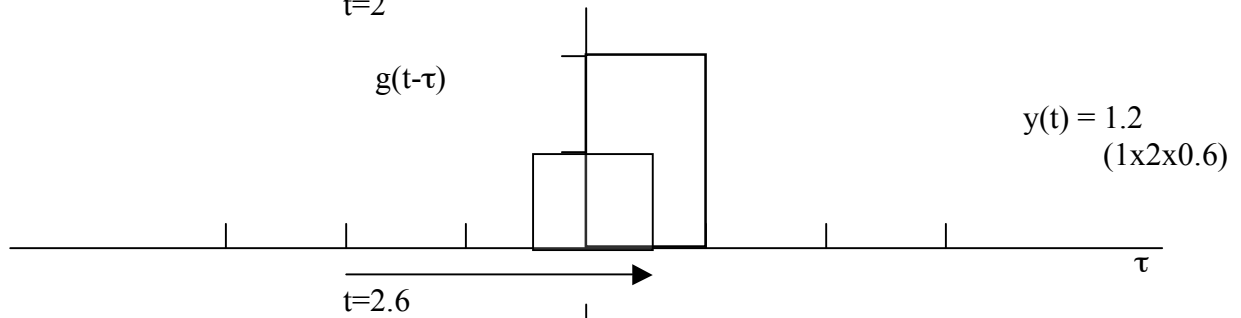
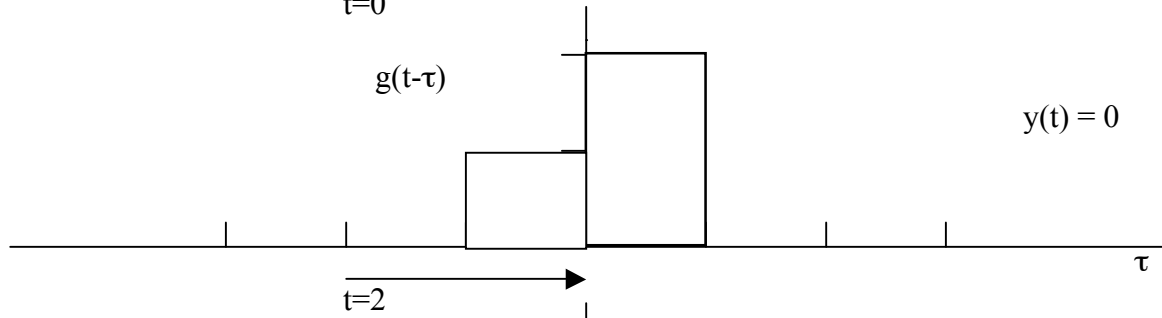
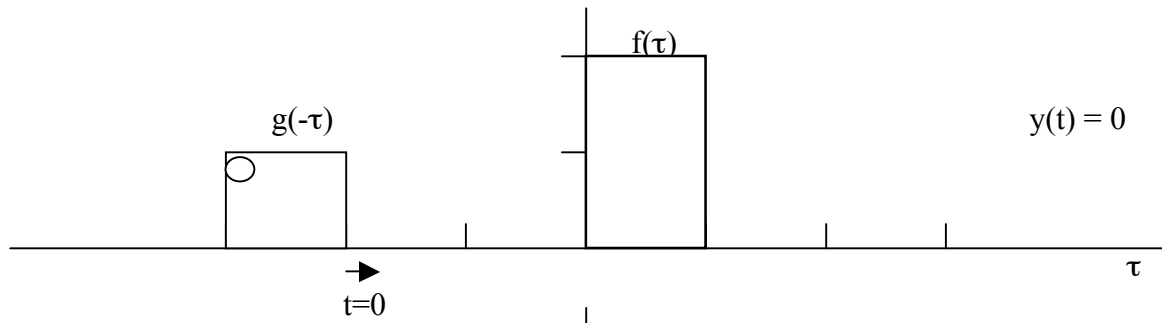
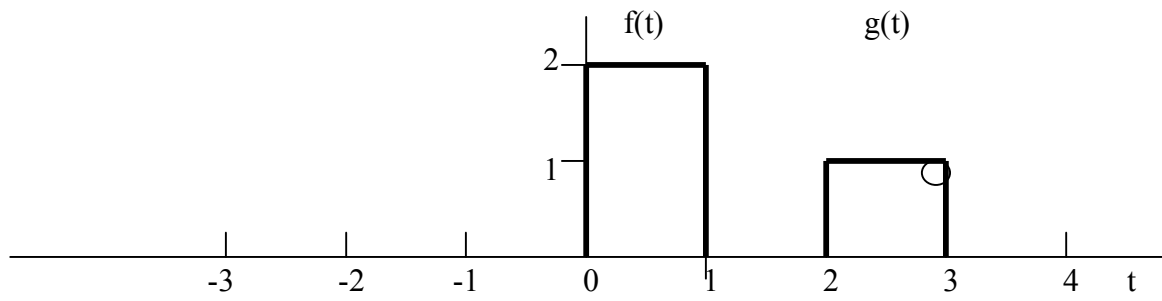
$$FT\{r(t) * h(t)\} = R(\omega)H(\omega)$$

$$FT\{r(t) \bullet h(t)\} = \frac{1}{2\pi} R(\omega) * H(\omega)$$

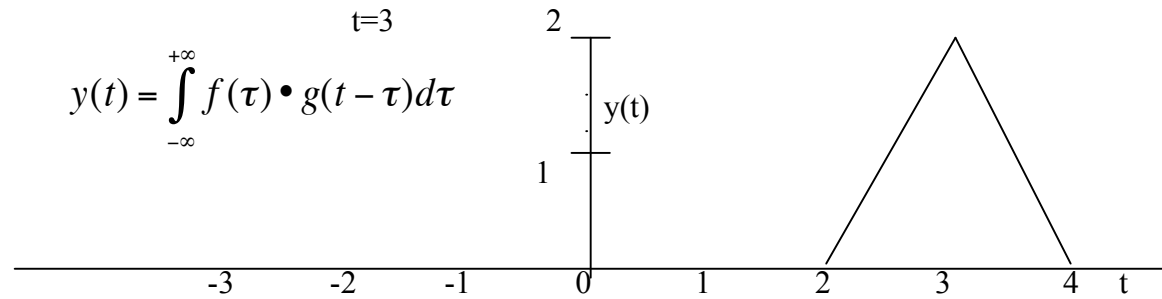
Example of a graphical convolution:

Given two gate functions, $f(t)$ and $g(t)$ as shown in the figure below. What is the convolution of $f(t)$ with $g(t)$, i.e. $y(t) = f(t)*g(t)$. In other words, what is

$$y(t) = \int_{-\infty}^{+\infty} f(\tau) \bullet g(t - \tau)d\tau \quad ?$$



$$y(t) = \int_{-\infty}^{+\infty} f(\tau) \cdot g(t - \tau) d\tau$$



Back to our example of the delay line with

$$r(t) = A \cos \omega_0 t$$

$$H(\omega) = B e^{-i\omega t_d}$$

$$y(t) = r(t) * h(t)$$

$$h(t) = FT^{-1} H(\omega)$$

$$= \frac{B}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega(t-t_d)} d\omega$$

$$= B \delta(t - t_d)$$

Then

$$y(t) = AB \int_{-\infty}^{+\infty} \cos(\omega_0 \tau) \delta(t - t_d - \tau) d\tau$$

To evaluate this integral we need to make use of the shifting and scaling properties of the δ -function (p. 16).

$$\int_{-\infty}^{+\infty} f(t) \delta(t - t_0) dt = f(t_0)$$

$$\int_{-\infty}^{+\infty} f(t) \delta(at) dt = \frac{1}{|a|} f(0)$$

$$y(t) = AB \cos[\omega_0(t - t_d)]$$

So we see, that we come to the same result as on p. 28.

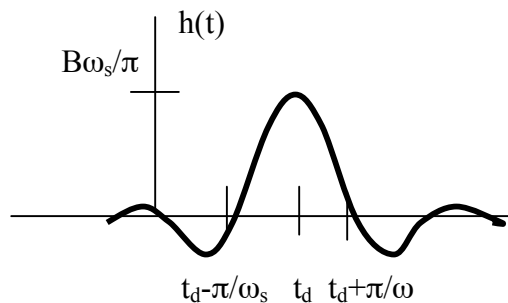
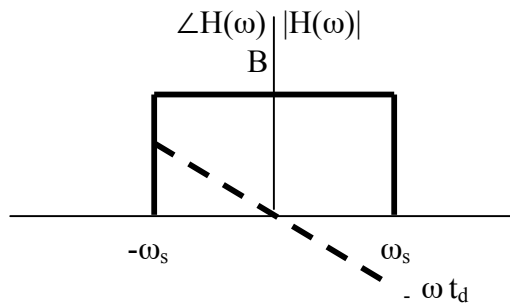
Ideal low-pass filters

$$H_{LPPF}(\omega) = \begin{cases} B e^{-i\omega t_d} & \text{for } |\omega| \leq \omega_s \\ 0 & \text{,elsewhere} \end{cases}$$

What is $h_{LPPF}(t)$?

$$h_{LPPF}(t) = \frac{1}{2\pi} \int_{-\omega_s}^{+\omega_s} B e^{i\omega(t-t_d)} d\omega$$

$$\begin{aligned}
&= \frac{B}{2\pi} \frac{1}{i(t-t_d)} \left(e^{i\omega(t-t_d)} \right) \Big|_{-\omega_s}^{+\omega_s} \\
&= \frac{B}{\pi} \frac{1}{(t-t_d)} \left[\frac{1}{2i} \left(e^{i\omega_s(t-t_d)} - e^{-i\omega_s(t-t_d)} \right) \right] \\
&= \frac{B}{\pi} \frac{1}{(t-t_d)} \sin \omega_s(t-t_d) \\
&= \frac{B}{\omega_s \pi} \frac{\sin \omega_s(t-t_d)}{\omega_s(t-t_d)}
\end{aligned}$$



Note: for $|\omega_s| \rightarrow \infty$ $h(t) \rightarrow B \delta(t-t_d)$

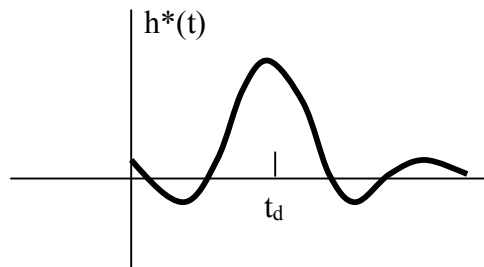
Physically realizable filters

For a physically realizable filter, $h(t)$ must be causal or non-anticipatory, i.e.

$h(t) = 0$ for $t < 0$.

To be physically realizable we must find filters with a transfer function

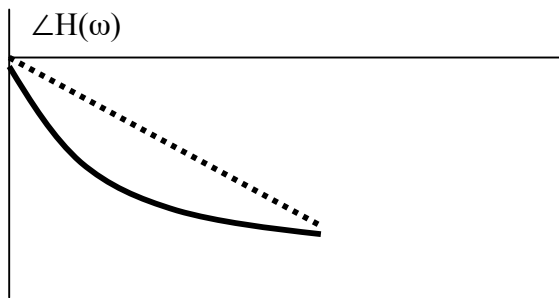
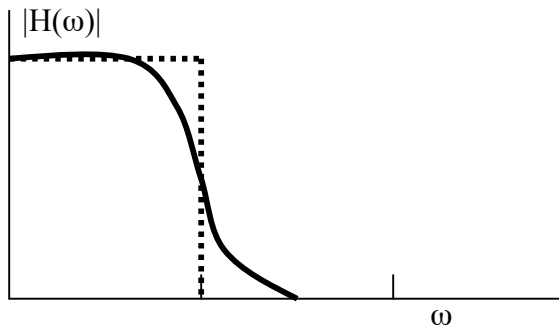
$h^*(t) = h(t)u(t)$.



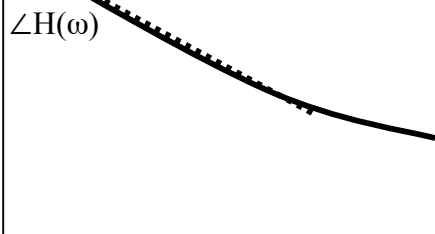
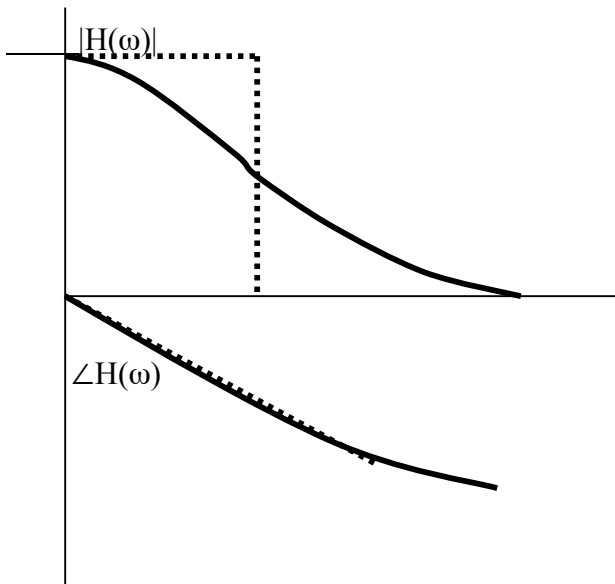
However, such a manipulation will distort $H^*(t)$. Distortions could be minimized if t_d is increased. For practical purposes, if $t_d \sim 2$ to 3 times π/ω_s , then $h^*(t)$ is a close version of $h(t)$. In radio technology, there are three examples of low-pass filters that minimize distortions and have principally different characteristics.

- Butterworth filter: constant gain
- Bessel filter: linear phase
- Chebyshev filter: good attenuation outside passband

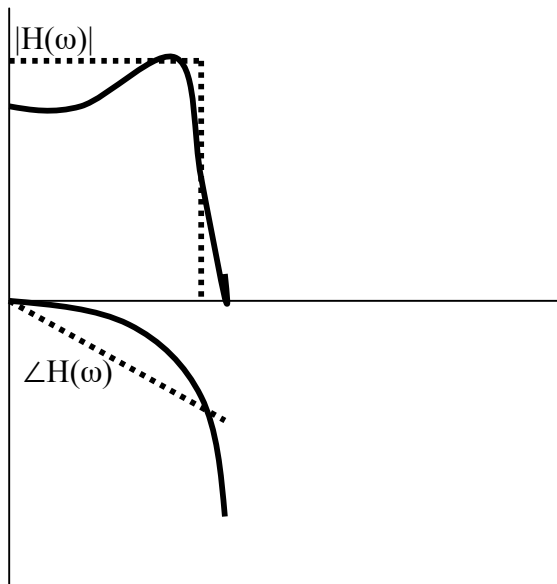
Butterworth filter



Bessel filter

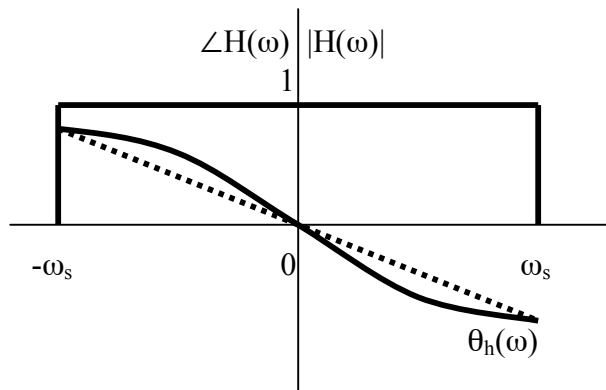


Chebyshev filter



Example of a signal distortion in a filter

Suppose we have a filter with the magnitude of the transfer function being ideal and with the phase of the transfer function being distorted.



$$|H(\omega)| = \begin{cases} 1, & \text{for } |\omega| \leq \omega_s \\ 0, & \text{for } |\omega| > \omega_s \end{cases}$$

$$\theta_h(\omega) = \begin{cases} -\omega t_d - k \sin \omega T, & \text{for } |\omega| \leq \omega_s \\ \text{undefined}, & \text{for } |\omega| > \omega_s \end{cases}$$

$$H(\omega) = |H(\omega)| e^{i\theta_h(\omega)}$$

$$H(\omega) = 1 e^{-i\omega t_d} e^{-ik \sin \omega T}, \dots \text{for } |\omega| \leq \omega_s, = 0 \text{ for } |\omega| > \omega_s$$

$$H(\omega) = e^{-i\omega t_d} e^{-ik \sin \omega T} \Pi\left(\frac{\omega}{2\omega_s}\right)$$

If the distortion is small so that $k \ll 1$, then we can write

$$\begin{aligned} e^{-ik \sin \omega T} &= 1 - ik \sin \omega T \\ &= 1 - ik \frac{1}{2i} (e^{i\omega T} - e^{-i\omega T}) \end{aligned}$$

What is then the output $y(t)$ of our filter for any arbitrary input $r(t)$?

$$y(t) = \text{FT}^{-1}\{R(\omega)H(\omega)\}$$

$$= \text{FT}^{-1}\left\{R(\omega)e^{-i\omega t_d} \left[1 - \frac{k}{2}(e^{i\omega T} - e^{-i\omega T})\right] \Pi\left(\frac{\omega}{2\omega_s}\right)\right\}$$

$$= \text{FT}^{-1}\left\{\left[R(\omega)e^{-i\omega t_d} - \frac{k}{2}R(\omega)e^{i\omega(t_d-T)} + \frac{k}{2}R(\omega)e^{i\omega(t_d+T)}\right] \Pi\left(\frac{\omega}{2\omega_s}\right)\right\}$$

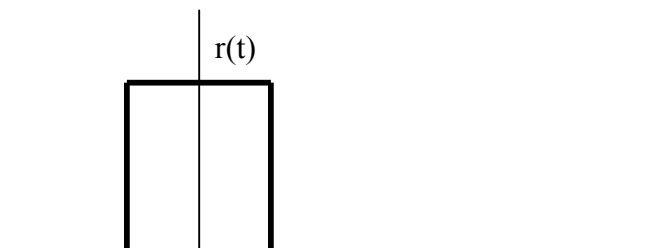
$$= \left[r(t-t_d) - \frac{k}{2}r(t-(t_d-T)) + \frac{k}{2}r(t-(t_d+T))\right] * \frac{1}{2\pi} 2\omega_s \frac{\sin \omega_s t}{\omega_s t}$$

(also see p. 33).

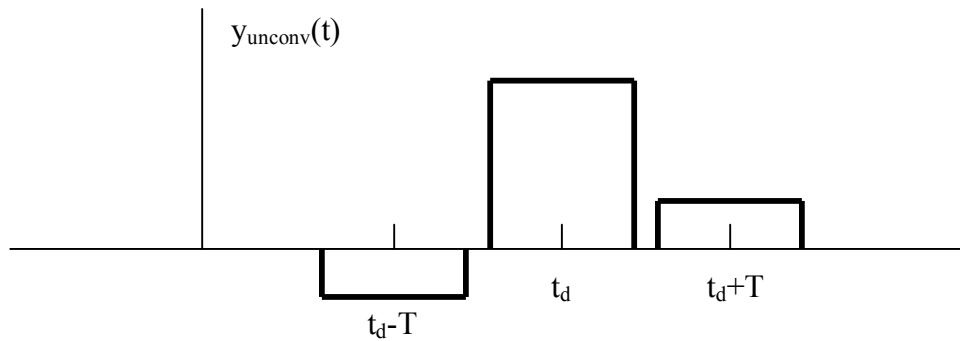
$$y_{\text{unconv}}(t) = \left[r(t-t_d) - \frac{k}{2}r(t-(t_d-T)) + \frac{k}{2}r(t-(t_d+T))\right]$$

$$y(t) = y_{\text{unconv}}(t) * \frac{1}{2\pi} 2\omega_s \frac{\sin \omega_s t}{\omega_s t}$$

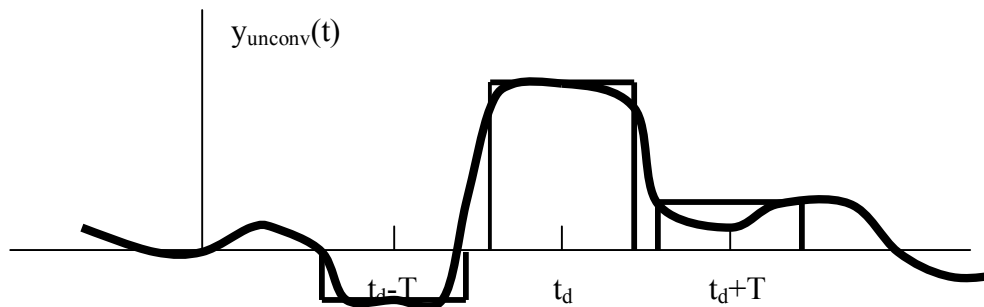
Let us now assume a particular form for $r(t)$, say as what is shown in the figure below.



Then, $y_{\text{unconv}}(t)$ will be



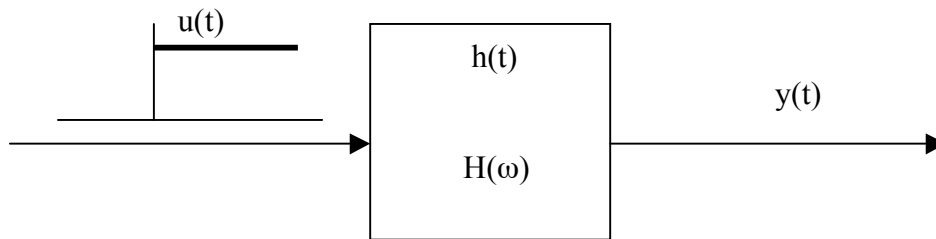
And, $y(t)$ will be (with an educated guess) approximately



Now imagine that you have a digitized message with a sequence of pulses. These echos pictured above could interfere with the leading and the trailing pulse and cause “intersymbol interference in communications systems. How can that be avoided? By using appropriate filters in networks that minimize distortions. Note: an ideal filter would not be necessarily appropriate either, since the delayed pulse would still be convolved with a sinc function that leads to “ringing.” What is used in practice are “raised cosine filters.”

Rise time of an ideal low-pass filter

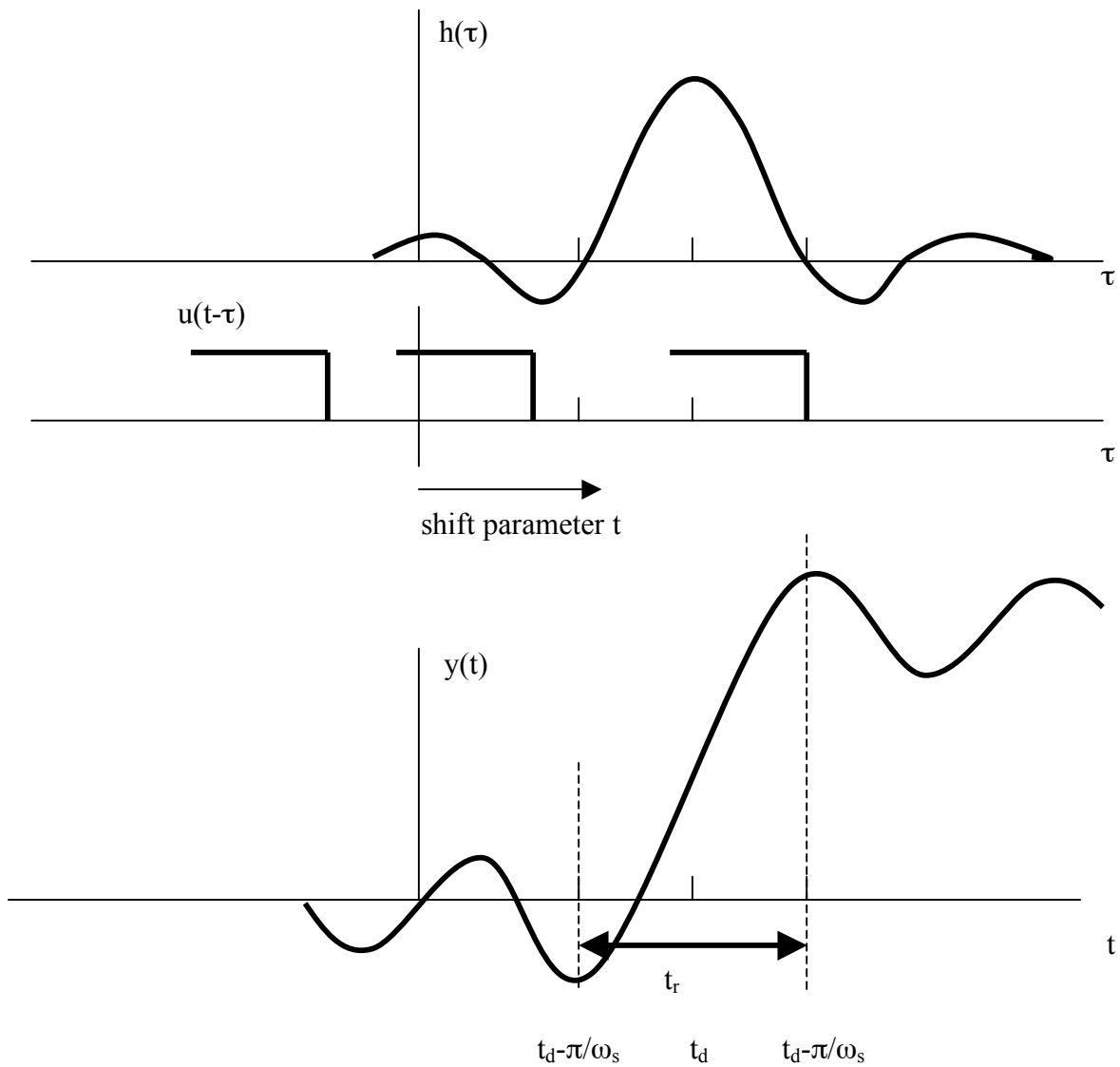
We measure the rise time of a filter by applying a step function to the filter input terminals and monitoring the output, $y(t)$



The rise time of the filter is then given by the time it takes $y(t)$ to go from its minimum to its maximum.

$$\begin{aligned}
 y(t) &= u(t) * h(t) \\
 &= h(t) * u(t) \\
 &= \int_{-\infty}^{+\infty} h(\tau) u(t - \tau) d\tau
 \end{aligned}$$

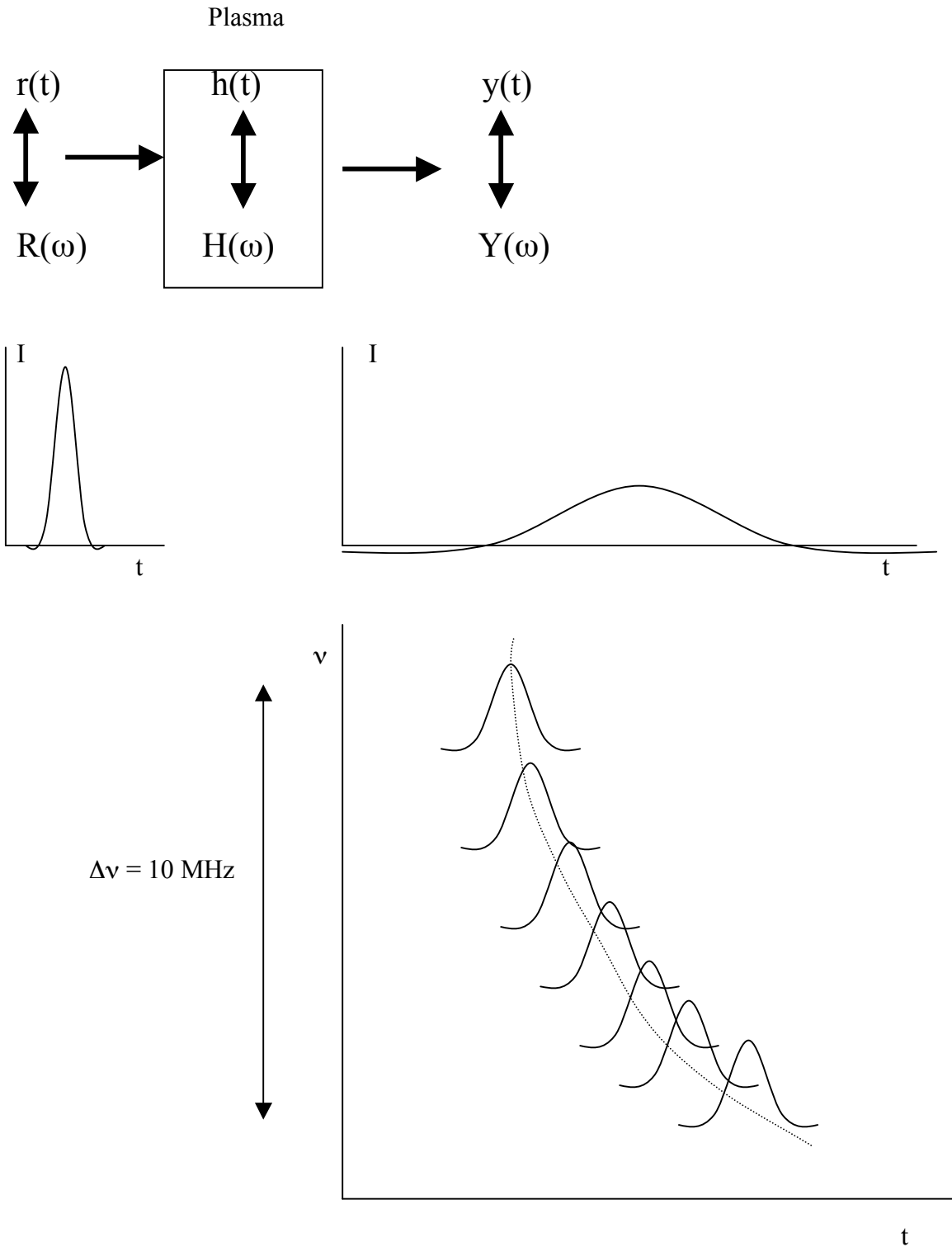
Let's do the graphical convolution.



$$\begin{aligned}
 t_r &= 2\pi/\omega_s \\
 &= 1/\Delta\nu
 \end{aligned}$$

Example: $\Delta\nu = 1 \text{ MHz}$
 $t_r = 1 \mu\text{s}$

Example: Dispersion of a pulse in a plasma

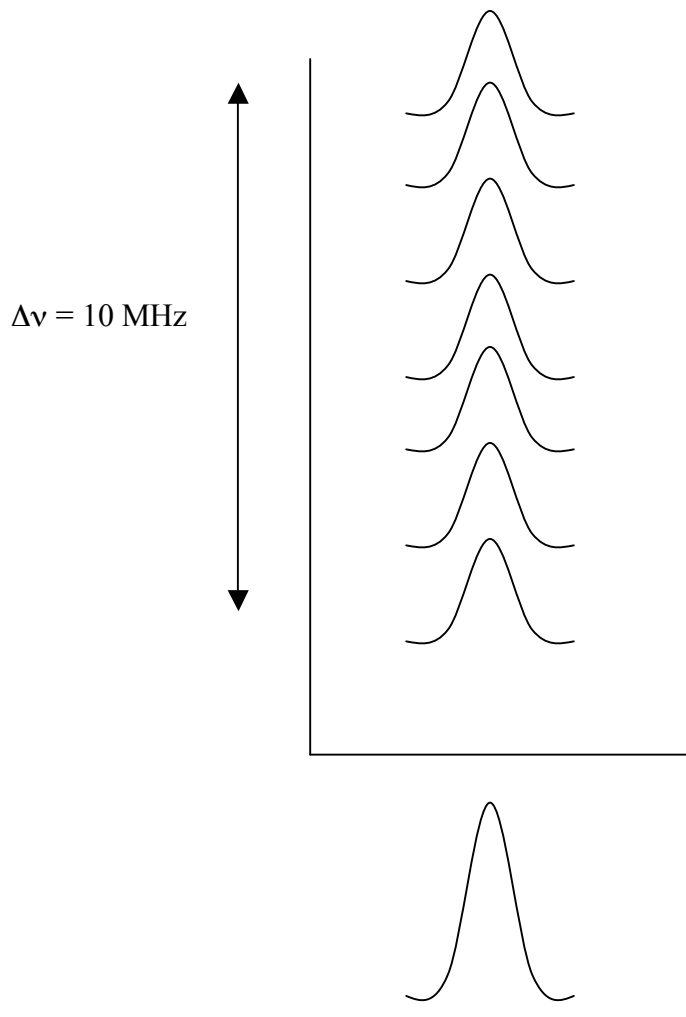


The plasma is dispersive meaning that the phase of its transfer function is a function of frequency, ν . That has the effect that pulses emitted at a higher frequency will arrive earlier and pulses at a lower frequency will arrive later. Now suppose that we have a radio antenna with a receiver that has a bandwidth $\Delta\nu$, say 10 MHz. Within this bandwidth, the pulse will arrive at a time dependent on frequency. For the ionosphere or

for interplanetary or interstellar space, the arrival time is approximately proportional to $1/v^2$. The received signal, $y(t)$, will therefore be a smeared-out version of $r(t)$. How can we recover $r(t)$?

There are two possibilities:

1. Use a filter bank. Each filter has a bandwidth, $\Delta\nu_i \ll \Delta\nu$ and a center frequency that is offset from the adjacent filters in such a way that the bandwidths of all the filters will just cover the bandwidth of the receiver. In effect we have chopped the receiver bandwidth of 10 MHz into smaller bandwidths, say 10 times 1 MHz. With such a filter bank we could record the output of the individual filters separately and shift the output of each filter in time so that the delay discrepancies are compensated for.



The result is a pulse that is dedispersed – but only to a certain degree. Why? Because the pulse from each filter of the filter bank is still dispersed, only over a bandwidth of 1 MHz as in our example, but still measurable? For instance, in our figure above, the pulse is still a bit smeared out.

How can we dedisperse the pulse completely? We could build a filter bank with a larger number of filters and narrow the bandwidth of the individual filters. Say, we could use 100 filters with each having a bandwidth of 100 kHz. Or we could build 1000 filters for

the filter bank with each filter having a bandwidth of just 10 kHz. However, the problem is that the narrower we choose the bandwidth, the longer the rise time of the individual filters become. If our original pulse, $r(t)$, has e.g. a duration of only $1 \mu\text{s}$, then a filter with a bandwidth of 10 kHz would broaden the pulse to the rise time of the filter which is $100 \mu\text{s}$. In this case, a filter bank would not be useful.

2. The second possibility is to dedisperse $r(t)$ directly.
 - Record $y(t) = r(t) * h(t)$, that is the signal $r(t)$ dispersed by the plasma
 - Compute $Y(\omega)$
 - Compute $Y(\omega) H^{-1}(\omega) = R(\omega)$
 - Compute $r(t)$

This technique has been used successfully for the dedispersion of ultra-short pulses from a neutron star. Pulses with a duration of less than 10 ns could be recovered. See www.yorku.ca/bartel and click on “in the news” under pulsars. Then click on New Scientist.

Dispersed signals are very common in radio science and radio technology. In radar technology, chirp filters are used for a variety of applications. These filters can be used as “compressors” as they concentrate energy into a narrow bandwidth. Other technologies worth mentioning are acousto-optic spectrometers which have also dispersive or dedispersive characteristics.

1.7 Energy, power and their spectral densities

Time-domain expressions

Total energy of a signal $f(t)$ in time interval $t_1 < t \leq t_2$

$$E = \int_{t_1}^{t_2} f(t) f^*(t) dt = \int_{t_1}^{t_2} |f(t)|^2 dt$$

Total energy of $f(t)$ over all time

$$E_{\infty} = \int_{-\infty}^{+\infty} f(t) f^*(t) dt = \int_{-\infty}^{+\infty} |f(t)|^2 dt$$

If the signal exists over the entire interval $(-\infty, +\infty)$, we need to use the power instead of the energy. The power, P , is defined as;

Power of $f(t)$ in time interval $t_1 < t \leq t_2$

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t) f^*(t) dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |f(t)|^2 dt$$

Power of $f(t)$ over all time:

$$P_{\infty} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f(t) f^*(t) dt$$

We now define:

Energy signal $f(t)$: $E_{\infty} < \infty$ ($\Rightarrow P_{\infty} = 0$)

Power signal $f(t)$: $P_{\infty} < \infty$ ($\Leftarrow E_{\infty} = \infty$)

Frequency-domain expressions for E_{∞} - Parseval's theorem

$$\begin{aligned} E_{\infty} &= \int_{-\infty}^{+\infty} f(t) f^*(t) dt \\ &= \int_{-\infty}^{+\infty} f(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} F^*(\omega) e^{-i\omega t} d\omega \right\} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F^*(\omega) \left\{ \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt \right\} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F^*(\omega) F(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega \end{aligned}$$

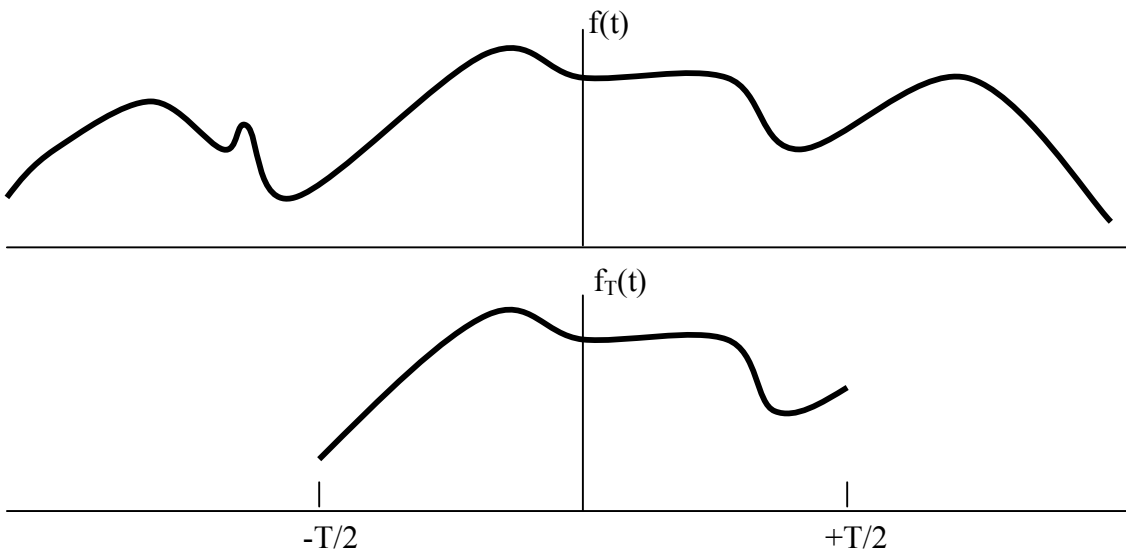
$|F(\omega)|^2$ is called: energy density spectrum or energy spectral density

Frequency-domain expressions for P_{∞}

Since for some signals, $E_{\infty} = \infty$, i.e. $\int_{-\infty}^{+\infty} |f(t)|^2 dt = \infty$, then, according to the Dirichlet

conditions, $f(t)$ is not necessarily Fourier transformable. Therefore we have to consider the truncated signal $f_T(t)$.

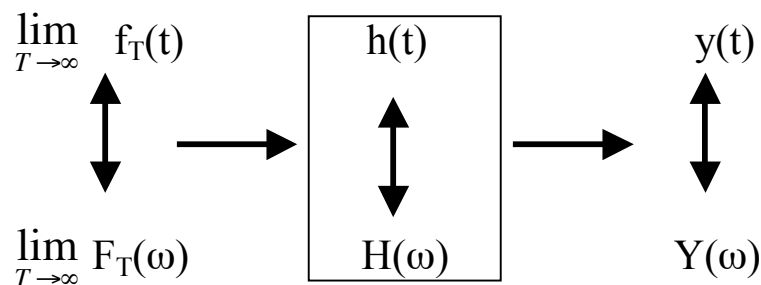
$$f_T(t) = \begin{cases} f(t), & |t| < T/2 \\ 0, & \text{elsewhere} \end{cases}$$



$$\begin{aligned}
 P_\infty &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f(t) f^*(t) dt \\
 &= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F_T(\omega)|^2 d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} |F_T(\omega)|^2 \right\} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_f(\omega) d\omega
 \end{aligned}$$

$S_f(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} |F_T(\omega)|^2$ is called power density spectrum or power spectral density (PSD)

How is the output power spectral density related to the input power spectral density?



$$\begin{aligned}
P_{\infty,y} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |y(t)|^2 dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y(\omega)|^2 d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |H(\omega)|^2 \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{+\infty} |F_T(\omega)|^2 d\omega
\end{aligned}$$

$$S_y(\omega) = |H(\omega)|^2 S_f(\omega)$$

Output power spectral density is equal to the magnitude of transfer function squared times the input power spectral density.

We have seen that E_∞ for a signal can be computed in the time domain as well as in the frequency domain. If we compute E_∞ in the frequency domain, we need the energy spectral density (ESD).

There are two ways to compute the ESD.

- 1) compute the FT of $f(t)$, namely $F(\omega)$ and then $|F(\omega)|^2$
- 2) compute the autocorrelation function $C_{ff}(t)$ and then its FT which is $|F(\omega)|^2$

$$\begin{array}{ccc}
f(t) & \longrightarrow & C_{ff}(t) \\
\updownarrow & & \updownarrow \\
F(\omega) & \longrightarrow & |F(\omega)|^2
\end{array}$$

autocorrelation function of $f(t)$: ACF

$$C_{ff}(t) = \int_{-\infty}^{+\infty} f^*(\tau) f(t + \tau) d\tau$$

Note: Sometimes the above function is called autocovariance function and the normalized autocovariance function, $\frac{C_{ff}(t)}{C_{ff}(0)}$, is then called the autocorrelation function.

Wiener-Khinchine Theorem:

$$C_{ff}(t) \leftrightarrow |F(\omega)|^2$$

There are two ways to compute the PSD.

- 3) compute the FT of $f_T(t)$, namely $F_T(\omega)$ and then $S_f(\omega)$.
- 4) compute the time-averaged autocorrelation function and then its FT which is $S_f(\omega)$.

$$\begin{array}{ccc} \lim_{T \rightarrow \infty} f_T(t) & \longrightarrow & \overline{C}_{ff}(t) \\ \updownarrow & & \updownarrow \\ \lim_{T \rightarrow \infty} F_T(\omega) & \longrightarrow & S_f(\omega) \end{array}$$

time-averaged autocorrelation function of f(t):

$$\overline{C}_{ff}(t) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f^*(\tau) f(t + \tau) d\tau$$

$$\overline{C}_{ff}(t) \leftrightarrow S_f(\omega)$$

The ACF has a number of important properties

1. For signals, $f(t)$ with a zero mean, $\bar{f} = 0$, with $\bar{f} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f(t) dt$, the

ACF at $t=0$ (zero lag) equals the signal variance:

$$\begin{aligned} \overline{C}_{ff}(0) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f^*(\tau) f(\tau) d\tau \\ &= \sigma_f^2 \end{aligned}$$

2. The ACF is hermitian:

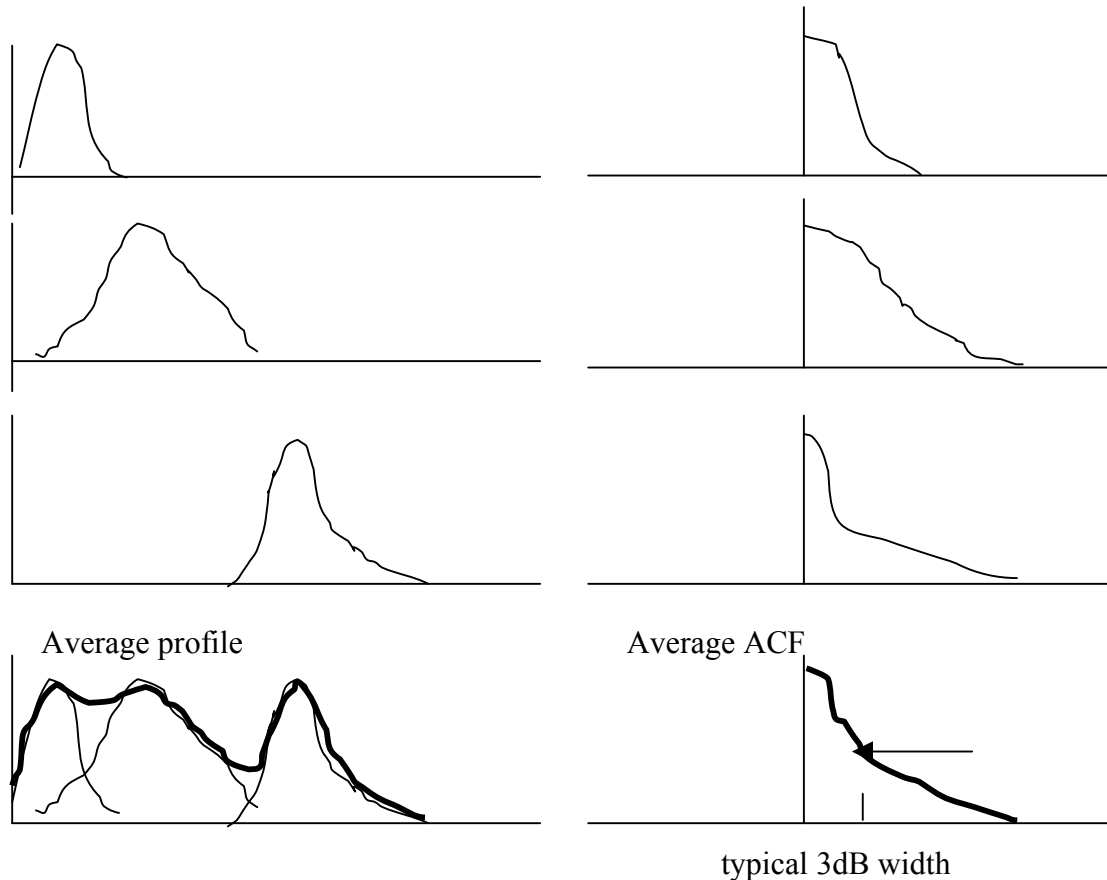
$$C_{ff}(-t) = C_{ff}^*(t)$$

$$C_{ff}(-t) = C_{ff}(t) \quad \text{for } f(t) : \text{ real}$$

3. If $f(t)$ has an ACF $C_{ff}(t)$, then $f(t-t_0)$ has also the ACF $C_{ff}(t)$, i.e. it is unchanged.

4. The normalized ACF, $\frac{C_{ff}(t)}{C_{ff}(0)}$ or $\frac{\overline{C}_{ff}(t)}{\overline{C}_{ff}(0)}$ is equal to 1 at $t=0$.

ACF's are widely used in radio sciences and radio technology - apart from being used for the computation of the ESD and the PSD. For instance, the ACF can be used to define statistically meaningful pulse widths. If for instance a pulse, as that given below, is to be analyzed in terms of a typical fluctuation duration, we could use the ACF.



If we have two complex functions, $f(t)$ and $g(t)$, we can define the cross-correlation function (CCF)

$$C_{fg}(t) = \int_{-\infty}^{+\infty} f^*(\tau)g(t + \tau)d\tau$$

and the time-averaged cross-correlation function

$$\bar{C}_{fg}(t) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f^*(\tau)g(t + \tau)d\tau$$

The normalized CCF can be defined as follows:

$$\frac{C_{fg}(t)}{C_{ff}(0)C_{gg}(0)} \quad \text{or} \quad \frac{\bar{C}_{fg}(t)}{\bar{C}_{ff}(0)\bar{C}_{gg}(0)}$$

As with the ACF we can write:

$$f(t), g(t) \longrightarrow C_{fg}(t)$$

$$\updownarrow \qquad \qquad \updownarrow$$

$$F(\omega), G(\omega) \longrightarrow F^*(\omega)G(\omega)$$

And similarly:

$$\lim_{T \rightarrow \infty} f_T(t), \lim_{T \rightarrow \infty} g_T(t) \longrightarrow \overline{C}_{fg}(t)$$

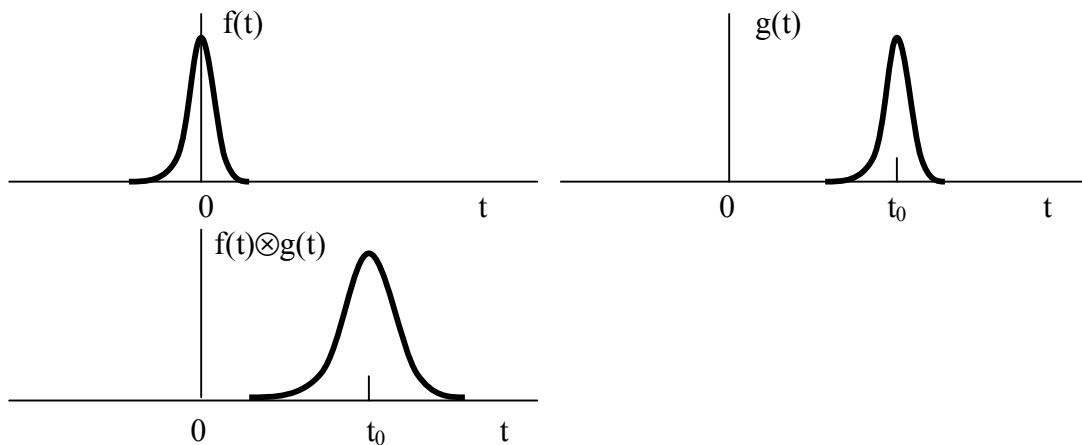
$$\updownarrow \qquad \qquad \updownarrow$$

$$\lim_{T \rightarrow \infty} F_T(\omega), \lim_{T \rightarrow \infty} G_T(\omega) \longrightarrow S_{fg}(\omega)$$

where $S_{fg}(\omega)$ is called the crosspower spectral density with

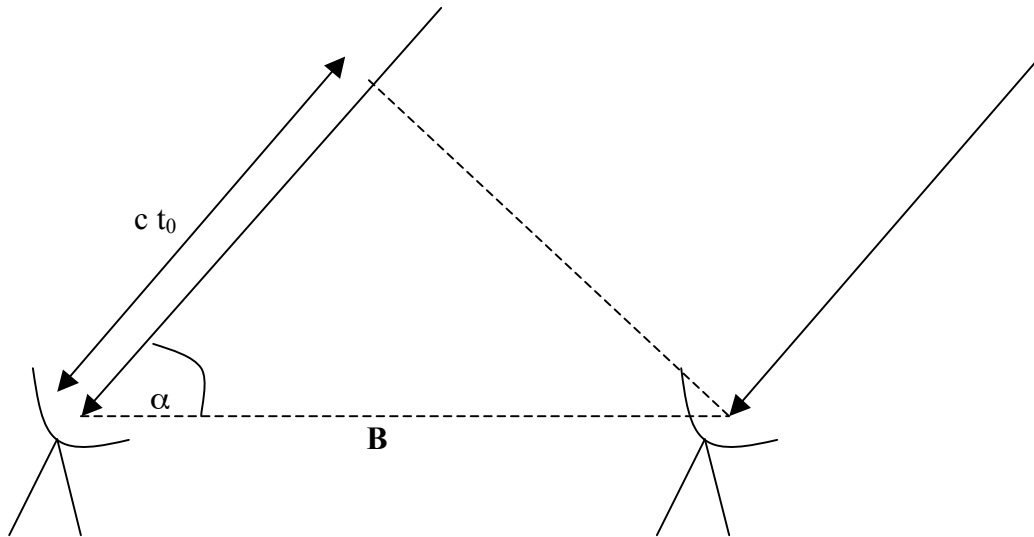
$$S_{fg}(\omega) = \lim_{T \rightarrow \infty} F_T^*(\omega)G_T(\omega)$$

CCF's are also widely used in radio sciences and radio technology - apart from being used for the computation of the functions above. For instance, the CCF can be used to determine the time lag between two similar signals.



Determining the lag or time delay between signals is for instance used in radio interferometry. An exact determination of the delay, t_0 , will lead to an exact determination of the position of an object in the sky provided the baseline vector, \mathbf{B} , is known.

Simple two-element interferometer:



Interferometers are used in astrometry, astronomy, geodesy, geophysics and spacecraft navigation.

Comparison between convolution and correlation

Convolution operator *

Correlation operator \otimes

Given two complex functions, $f(t)$ and $g(t)$.

$$f(t) \otimes g(t) = \int_{-\infty}^{+\infty} f^*(t + \tau)g(\tau)d\tau$$

with $\tau = -\tau'$, and $d\tau = -d\tau'$ we get

$$f(t) \otimes g(t) = \int_{-(-\infty)}^{-(+\infty)} f^*(-\tau')g(t - \tau')(-d\tau')$$

$$\begin{aligned} &= \int_{-\infty}^{+\infty} f^*(-\tau')g(t-\tau')d\tau' \\ &= f^*(-t) * g(t) \end{aligned}$$

For real functions, $f(t)$, $g(t)$

$$f(t) \otimes g(t) = f(-t) * g(t)$$

For real functions, $f(t)$, $g(t)$, with at least one function being symmetrical

$$f(t) \otimes g(t) = f(t) * g(t)$$