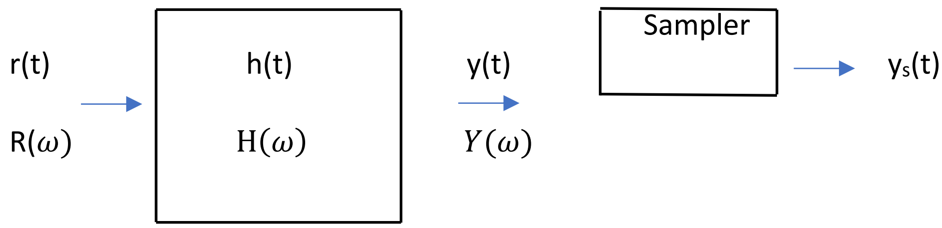
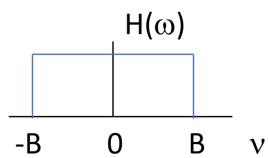
1.8 The sampling theorem

The rise time of a filter is closely related to the sampling theorem. Assume that we have an ideal lowpass linear filter with bandwidth, B, and an output signal y(t), and sample y(t) at a sampling frequency, ν_s to get the sampled signal $y_s(t)$.





(We will not consider the phase)

y(t) can be fully recovered if the sampling frequency, ν_s, is equal or greater than 2B

Nyquist frequency: $\nu_N = 2B$

Sampling theorem:

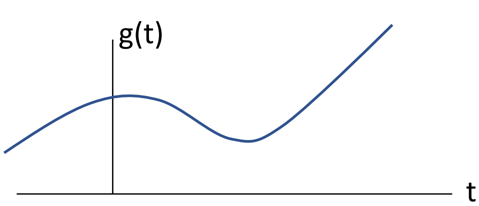
Any bandlimited function, g(t) with

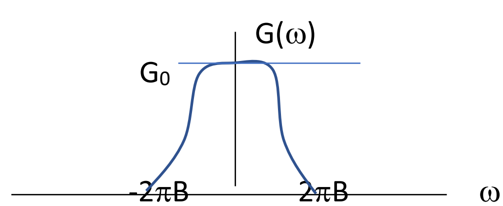
$$FT\{g(t)\} = \begin{cases} G(\omega), & |\omega| \leq 2\pi B \\ 0, & \text{elsewhere} \end{cases}$$

Can be recovered exactly from its samples taken at a rate of 2B samples/ sec or faster.

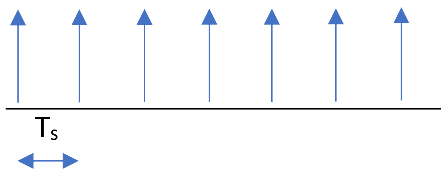
Proof:

Given: $g(t) \leftrightarrow G(\omega)$ with $G(\omega) = 0$ for $\omega \geq 2B$

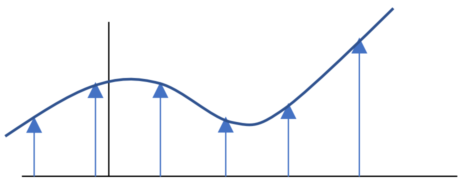




Sampling g(t) every T_s seconds means multiplying g(t) by a unit impulse train, or a train of δ functions.



$$\sum_{-\infty}^{\infty} \delta\,(t - nT_s)$$



$$g_s(t) = g(t) \sum_{-\infty}^{\infty} \delta\,(t - nT_s)$$

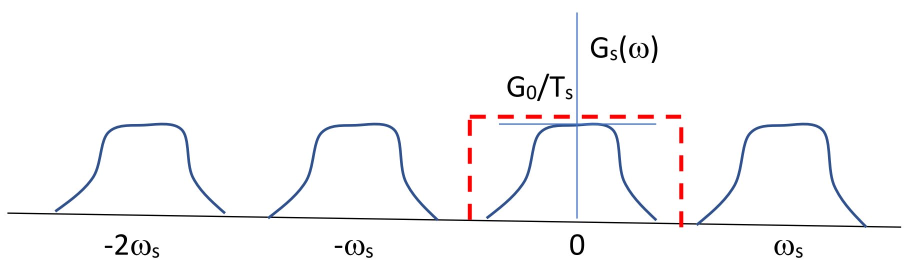
Using the FS for the impulse train gives us:

$$\sum_{-\infty}^{\infty} \delta\,(t - nT_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{in\omega_s t}$$

$$g_s(t) = g(t)\, \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{in\omega_s t}$$

$$g_s(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} g(t) e^{in\omega_s t}$$

$$G_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(\omega - n\omega_s)$$

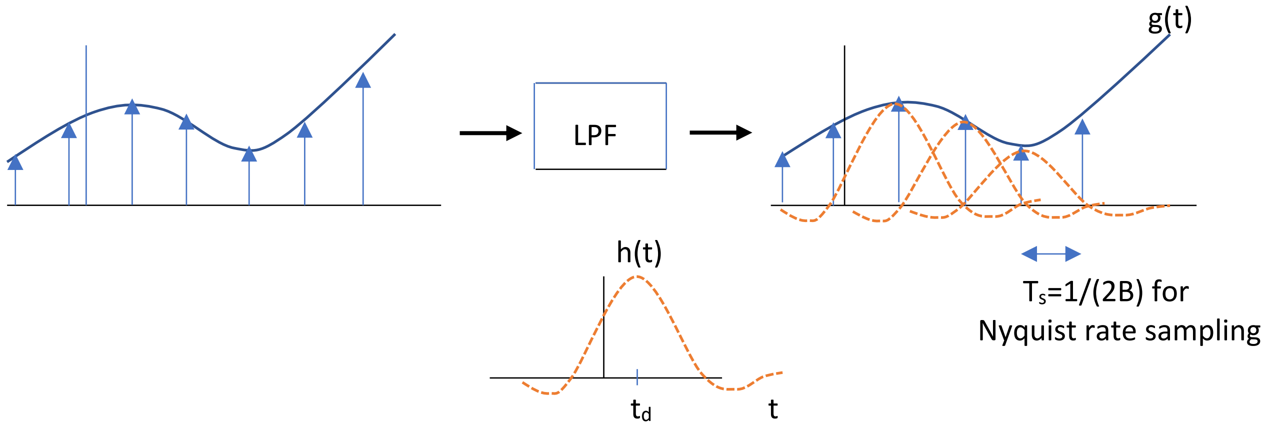


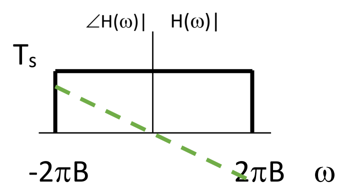
Now we need to use a low-pass filter $G_s(\omega)$ so that all lobes but the center lobe are rejected. Then FT^{-1} $\{1/T_s\, G(\omega)\} = 1/T_s\, g(t)$ and we have recovered g(t) from $g_s(t)$ with the condition that there is no overlap of the adjacent spectra.

Condition:

$$\omega_s = \frac{2\pi}{T_s} \geq 4\pi B \ , \qquad \frac{1}{T_s} \geq 2B \ , \qquad T_s \leq \frac{1}{2B}$$

How does the recovering process look in the time domain?





$$h_{LPF} = 2T_sB \cdot sinc[2B(t - t_d)]$$

$$g(t) = \int_{-\infty}^{\infty} g_s(t) \cdot h(t - \tau)d\tau$$

$$g(t) = \int_{-\infty}^{\infty} g(t) \sum_{-\infty}^{\infty} \delta\,(t - nT_s) \cdot h(t - \tau)d\tau$$

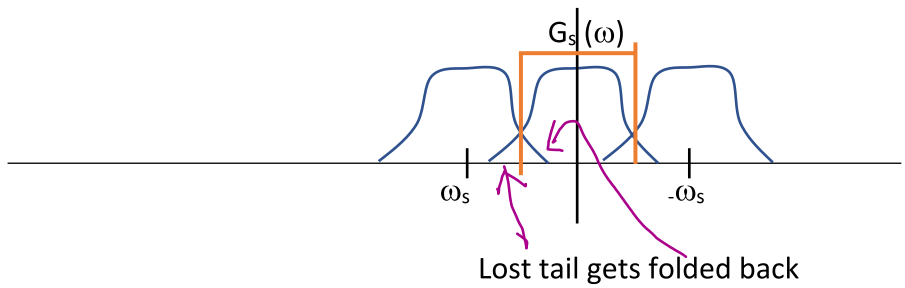
$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \cdot h(t - nT_s)$$

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \cdot 2T_sB \cdot sinc[2B(t - nT_s - t_d)]$$

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \cdot sinc[2B(t - nT_s)]$$

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \cdot sinc[2Bt - n]$$

What happens if there is an overlap of the spectra $G_s(\omega)$ with $T_s > 1/(2B)$?



It is not possible to recover g(t) from $g_s(t)$ by any means.

If $g_s(t)$ is passed through a low-pass filter, we get a spectrum that is a distorted version of $G(\omega)$ because:

1) Loss of the tail of $G(\omega)$ beyond $|\omega| \geq ½ \omega_s$
2) Inversion of folding of the same tail onto the spectrum at the cutoff frequency.

<u>Tail inversion: spectral folding or aliasing</u>

Question:

You film a Formula I racing car passing by and see the wheels going backwards. How can that be explained?