AP/ECON 2300 FF   Answers to Assignment 2   November 2010

Q1. If a person earned Y_P when young, and Y_F when old, how would her saving vary with the net rate of return r to saving, if her preferences could be represented by the utility function

$$u(C_P, C_F) = C_P^2 C_F$$

where C_P is her consumption when young and C_F her consumption when old?

A1. These preferences are examples of *Cobb–Douglas* preferences. The appendix to chapter 5 in Varian derives the demand function for Cobb–Douglas preferences : demand for good 1 is

$$x_1 = \frac{c}{c+d}\frac{m}{p_1} \tag{1-1}$$

if the preferences take the form $U(x_1, x_2) = x_1^c x_2^d$.

The problem described in the question [maximization of $U(C_P, C_F)$ when facing an interest rate of r, with income of Y_i in period i] can be written as in equation $(10-2)$ of Varian, as maximizing $U(C_P, C_F)$ subject to the budget constraint

$$(1+r)C_P + C_F = (1+r)Y_P + Y_F \tag{1-2}$$

which is a "standard" budget problem, in which the price of good P is $(1+r)$, the price of good F is 1, and the person's total lifetime income (expressed in period–F prices) is

$$m = (1+r)Y_P + Y_F \tag{1-3}$$

So with Cobb–Douglas preferences, the demand for consumption in period P is (from equation $(1-1)$)

$$C_P = \frac{c}{c+d}\frac{(1+r)Y_P + Y_F}{1+r} \tag{1-4}$$

since here $p_1 = 1+r$, and $m = (1+r)Y_P + Y_F$. Since the utility function is

$$U(C_P, C_F) = (C_P)^2 C_F \tag{1-5}$$

here $c = 2$ and $d = 1$ so that equation $(1-4)$ is

$$C_P = \frac{2}{3}(Y_P + \frac{Y_F}{1+r}) \tag{1-6}$$

[Equation $(1-6)$ can also be obtained directly, from maximization of $(C_P)^2 C_F$ subject to the constraint that $C_F = Y_F + (1+r)(Y_P - C_P)$.] Savings is just first–period income minus first–period consumption, so that

$$S = Y_P - C_P$$

which (from equation $(1-6)$) means that

$$S = \frac{1}{3}Y_P - \frac{2}{3}\frac{Y_F}{1+r} \tag{1-7}$$

Equation $(1-7)$ says that her level S of saving will be **independent** of the interest rate r if she expects to earn no income when old, and that the level of saving will **increase** with the interest rate if she expects to earn a positive amount of income when old ($Y_F > 0$).

Q2. A firm is considering setting up a mining operation. It estimates that it will cost 100 million dollars to set up the mine this year, and a further 126 million dollars to clean up the mine site when the mining is finished, two years from now. The mine is expected to earn to yield ore with a net value of 225 million dollars, which will all be extracted one year from now.

For what values of the annual interest rate would it be profitable for the firm to set up this mine?

A2. The present value of the mining operation, in millions of dollars, is

$$PV = -100 + \frac{225}{1+r} - \frac{126}{1+r}^{2} \tag{2-1}$$

The operation will be profitable if and only if the expression in equation $(2-1)$ is positive.

Whether this is positive or not depends on the interest rate r. For example, if the interest rate were 0, then $PV = -1$, so that the mine is not profitable. If the interest rate is very high, this present value must also be negative. (For example, if $r = 2$, then $PV < -100 + 225/3 = -25$, even if the clean–up costs are ignored.)

But for "medium" values of r, the mine will be profitable.

To find exactly which values, let

$$R = 1 + r$$

and multiply both sides of equation $(2-1)$ by R^2, so that

$$-R^2 PV = 100R^2 - 225R + 126 \tag{2-2}$$

To find what values of R make the right side of equation $(2-2)$ equal to zero, use the quadratic formula. The right side of equation $(2-2)$ will equal zero if and only if

$$R = -\frac{225}{200} \pm \frac{\sqrt{(225)^2 - 4(126)(100)}}{200} \tag{2-3}$$

or

$$R = \frac{225 \pm 15}{200} \tag{2-4}$$

with solutions $R = 1.2$ and $R = 1.05$.

So if the interest rate is between 5% and 20%, then the mining operation will have a positive present value.

The figure below illustrates the present value, as a function of the interest rate r used to discount the future.

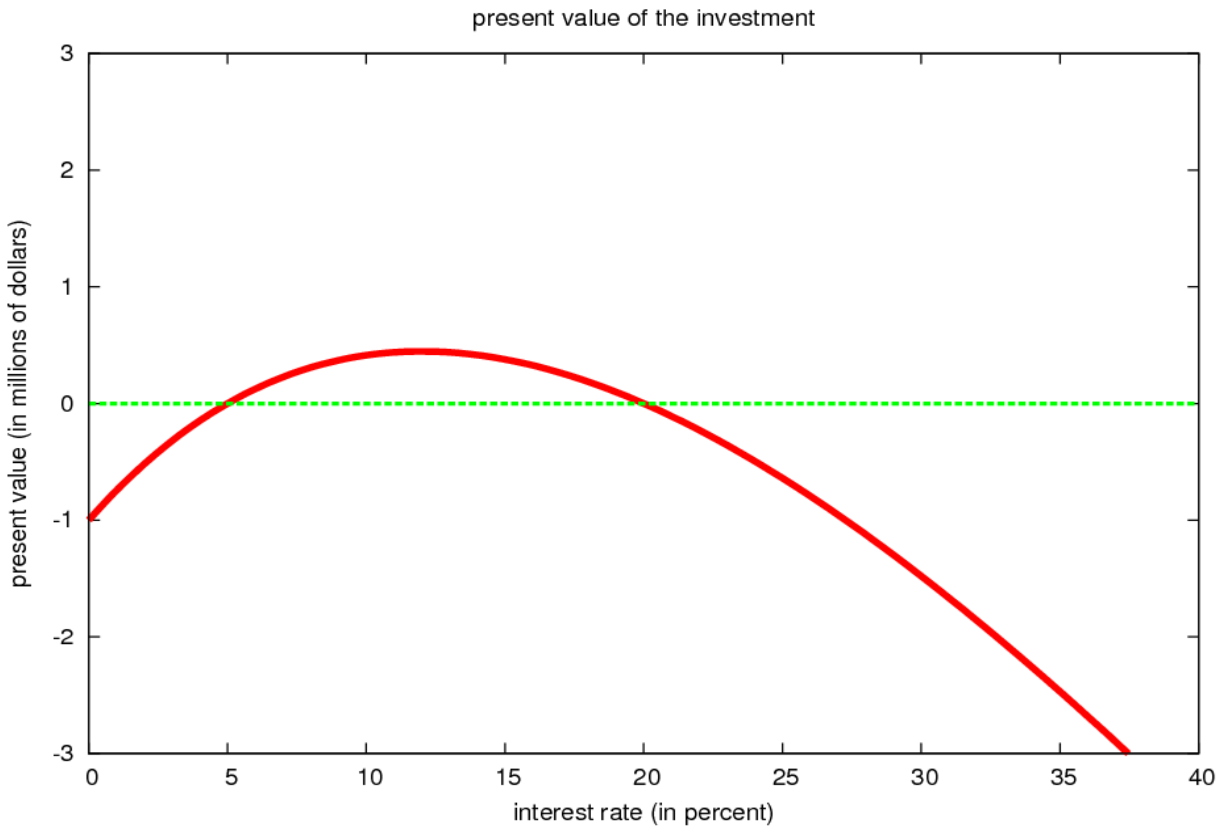

Figure (Question 2) : present value of profits, as a function of the interest rate r

Q3. A student is considering how much education to acquire. She has calculated that the present value of a person's lifetime earnings is defined by the following table, when the interest rate is 3% per year. [So if she were to stay in school for 3 years, she would earn a lifetime income, in present value, of $2,500,000, starting 3 years from now.]

If she wishes to maximize the net present value of her lifetime earnings (measured from today), how long should she stay in school, if the interest rate is 3% per year?

years schooling	PV of earnings
0	2,000,000
1	2,200,000
2	2,360,000
3	2,500,000
4	2,600,000
5	2,680,000
6	2,750,000
7	2,810,000
8	2,840,000
9	2,850,000
10	2,850,000

A3. Viewed from right now, the present value of a person's lifetime income is

$$W_t \equiv V_t/(1+r)^t \qquad (3-1)$$

if she stays in school for t years, where V_t is the value of her lifetime earnings, viewed from year t, if she stayed at school for t years (so that $V_3 = 2,500,000$, for example), and $r = 0.3$ since the interest rate is 3 percent per year.

As long as V_t grows faster than 3 percent, then she should stay in school. Why? Suppose that

$$V_{t+1} > (1.03)V_t \qquad (3-2)$$

Then

$$W_{t+1} = V_{t+1}/(1.03)^{t+1} = \frac{V_{t+1}}{(1.03)(1.03)^t} > \frac{(1.03)V_t}{(1.03)(1.03)^t} = \frac{V_t}{(1.03)^t} = W_t \qquad (3-3)$$

Similarly, if $V_{t+1} < (1.03)V_t$, then staying in school for another year would actually decrease W_t : the increase in V_t does not overcome the decrease in present value caused by having to wait a year longer before entering the work force.

In the example, V_t increases by 10% if she goes to school for 1 year, a further $(2.36 - 2.20)/(2.20) \approx 7.2$ percent if she stays in school for a second year, but the rate of return to education V_{t+1}/V_t keeps falling as she stays longer and longer in school.

The table below shows the rate of return V_{t+1}/V_t from staying in school 1 more year after year t.

years schooling	PV of earnings	return
0	$2,000,000$	10%
1	$2,200,000$	7.3%
2	$2,360,000$	5.9%
3	$2,500,000$	4%
4	$2,600,000$	3.07%
5	$2,680,000$	2.61%
6	$2,750,000$	2.2%
7	$2,810,000$	1.1%
8	$2,840,000$	0.3%
9	$2,850,000$	0%
10	$2,850,000$	

So she should stay in school for 5 years : the fifth year of schooling has a rate of return of $(2.68 - 2.6)/(2.6) \approx 3.07\%$, but the sixth year of schooling has a return of less than 3 percent (as do all subsequent years).

Q4. If a person's preferences could be represented by the utility function

$$u(F, C) = FC^2$$

where F is her food consumption and C her clothing consumption, and if the price of food were 1, and her income was 60, what would be the equivalent variation to a fall in the price of clothing from 2 to 1/2?

A4. These are again Cobb–Douglas preferences. So (using the appendix to chapter 5), the demand for clothing can be written (from equation $(1-1)$ above)

$$C = \frac{2}{3}\frac{m}{P_C} \tag{4-1}$$

Initially, $m = 60$ and $P_C = 2$, so that the person buys

$$C = \frac{2}{3}\frac{60}{2} = 20 \tag{4-2}$$

units of clothing, and 20 units of food, giving her an initial utility level of

$$U^0 = (20)(20)^2 = 8000 \tag{4-3}$$

If the price of clothing fell from 2 to 1/2 then

$$C = \frac{2}{3}\frac{60}{0.5} = 80 \tag{4-4}$$

and her food consumption remains at 20, so that her utility level would rise to

$$U^1 = (20)(80)^2 = 128000 \tag{4-5}$$

Finding the equivalent variation means finding the amount of money EV, so that the person would get a utility level equal to U^1, if, instead of lowering the price of clothing, we just gave her EV dollars.

So what would the person do if her income increased to $60 + EV$, but the prices stayed at their original levels, 1 for food and 2 for clothing?

Her demand for clothing would be

$$C = \frac{2}{3}\frac{60 + EV}{2} \qquad (4-6)$$

and her demand for food would be

$$F = \frac{1}{3}60 + EV \qquad (4-7)$$

From equations $(4-6)$ and $(4-7)$, giving her EV dollars would result in her having a utility level of

$$U = [\frac{1}{3}(60 + EV)][\frac{1}{3}(60 + EV)]^2 \qquad (4-8)$$

The equivalent variation is the level of EV for which $U = U^1$.

From equations $(4-5)$ and $(4-8)$, we want

$$\frac{1}{27}[60 + EV]^3 = 128000 \qquad (4-9)$$

or

$$(\frac{60 + EV}{3})^3 = 128,000 \qquad (4-10)$$

which means that

$$EV \approx 91.19 \qquad (4-11)$$

The figure below illustrates : pivoting the budget line out to the dotted purple budget line, by lowering P_C from 2 to 1/2 gets the person to the dotted green indifference curve ; so does shifting the budget line out parallel, to the dashed light blue line. So the amount of money needed to shift the dooted dark blue line out parallel, to the dashed light–blue line, is the equivalent variation to the clothing price decrease. Since the price of food is 1, the horizontal intercept measures the person's income. The parallel shift increases her income from 60, to just over 150 : the amount of the shift is about 91.

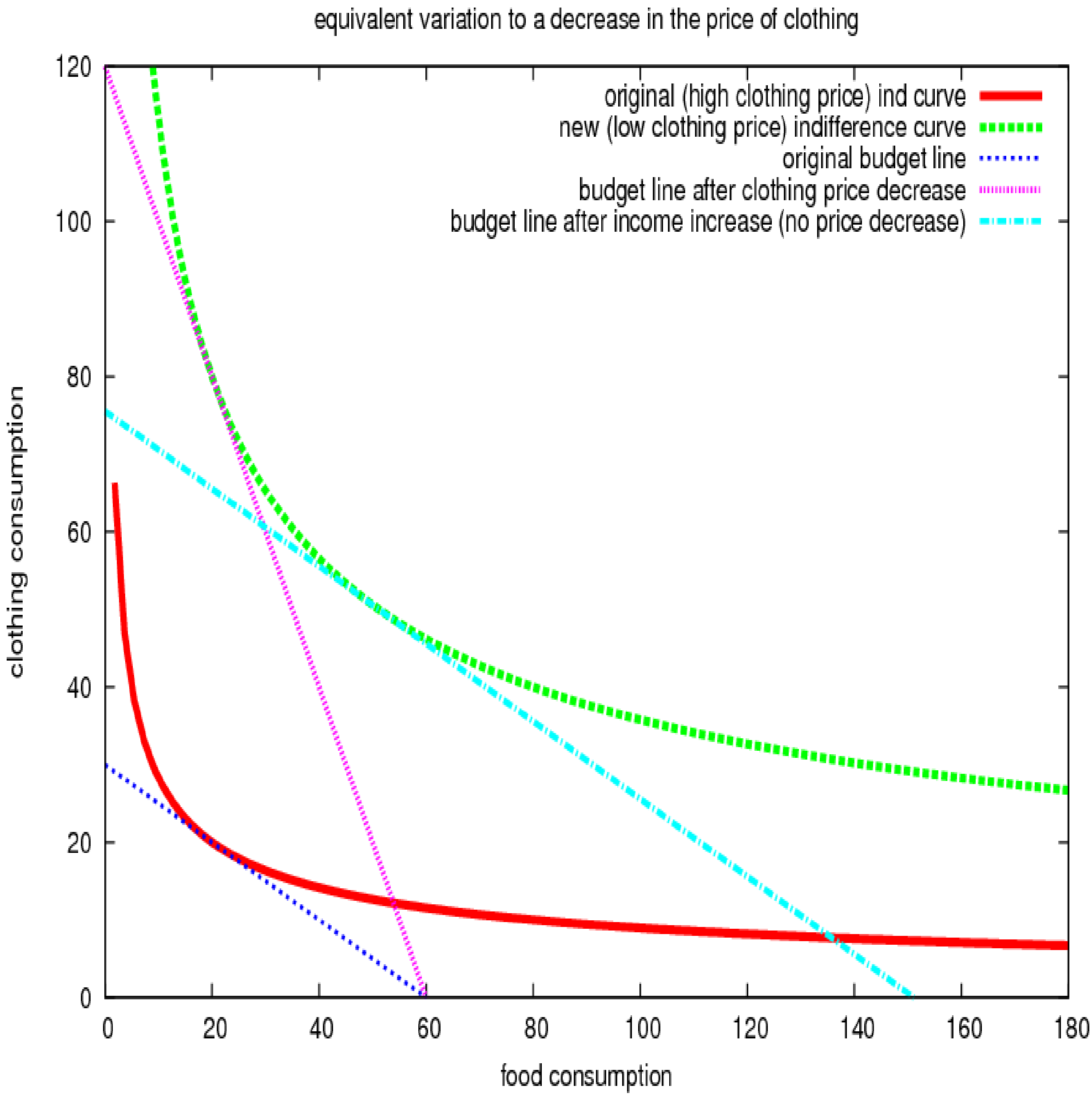

Figure (Question 4) : the person originally is on the red indifference curve ; a fall in the price of clothing or a gift of \$91.19 are equivalent, in that they both move her out to the dotted green indifference curve

Q5. A risk–averse expected utility maximizer has a utility function

$$u(W) = \sqrt{W}$$

She currently has wealth of 16 million dollars.

She has to choose between 2 alternative investments : she can only choose one of the two.

Investment #1 is risky. If it succeeds, she will quadruple her wealth. If it fails, she will lose 12 million dollars (from her original wealth of 16 million). She estimates that either outcome, success of failure, is equally likely.

Investment #2 is a sure thing, which will increase her wealth from 16 million dollars to X dollars, for sure.

If she is indifferent between the two investments, what must X be?

A5. If she undertakes investment #1, there is a fifty percent chance that her wealth will quadruple, from 16 million to 64 million. But there is also a fifty percent chance that her wealth will fall to 4 million.

So her expected **utility**, if she undertakes investment #1 is

$$EU_1 = 0.5\sqrt{64000000} + (0.5)\sqrt{4000000} = 0.5(8000) + 0.5(2000) = 5000 \qquad (5-1)$$

On the other hand, if she undertakes investment #2, her wealth will be X for sure, so that her expected utility will be

$$EU_2 = \sqrt{X} \qquad (5-2)$$

She will be indifferent between the investments if they both give her the same expected utility. So she will be indifferent between the investments if

$$\sqrt{X} = 5000 \qquad (5-3)$$

which is the same thing (squaring both sides of equation $(5-3)$) as

$$X = 5000^2 = 25000000 \qquad (5-4)$$

So an investment which is sure to increase her wealth from 16 million to 25 million is just as good (but no better) for her as a risky investment which increases her wealth to 64000000 with probability 0.5 and decreases it to 4 million with probability 0.5.