AP/ECON 2350 Answers to Assignment 1 W 2010–11

*Q*1. Give an example of a production function with increasing returns to scale, in which the inputs are perfect substitutes.

*A*1. The simplest production function in which the input are perfect substitutes is probably a production function like $f(x_1, x_2) = x_1 + 2x_2$. The problem is : this production function has *constant returns to scale*. Why? Here $f(tx_1, tx_2) = tx_1 + 2(tx_2) = t[x_1 + 2x_2] = tf(x_1, x_2)$.

So the way to make a perfect–substitutes production function have increasing returns to scale is to make the output increasing more rapidly with the inputs, by taking a monotonic transformation of the original production function. So we want a new production function $g(x_1, x_2) = \Psi(f(x_1, x_2))$, where $\Psi(y)$ is an increasing function, with $\Psi''(y) > 0$.

For example, let $\Psi(y) = y^2$, so that the new production function is $g(x_{1,2}) = (x_1 + 2x_2)^2$. This new production function still has the inputs being perfect substitutes. In this example

$$MP_1 = \frac{\partial g}{\partial x_1} = 2(x_1 + 2x_2)$$

$$MP_2 = \frac{\partial g}{\partial x_2} = 4(x_1 + 2x_2)$$

so that

$$TRS = \frac{MP_1}{MP_2} = \frac{1}{2}$$

Since the slope of each isoquant $(-TRS)$ is a constant, $1/2$, each isoquant is a straight line, so that we still have perfect substitutes with the production function $g(x_1, x_2)$.

What are the returns to scale? Here

$$g(tx_1, tx_2) = (tx_1 + 2tx_2)^2 = t^2(x_1 + 2x_2)^2 = t^2 g(x_1, x_2)$$

Since $t^2 > t$ if the scale t by which we increase the inputs is greater than 1, therefore $g(tx_1, tx_2) > tg(x_1, x_2)$ so that the production function $g(x_1, x_2) = (x_1 + 2x_2)^2$ exhibits increasing returns to scale.

*Q*2. What is the profit–maximizing output for a firm in perfect competition, if its production function is

$$y = \ln(x_1 + 1) + \ln(x_2 + 1)$$

(where ln refers to the natural logarithm function)?

*A*2. Given this production function, the perfectly competitive firm's profits are

$$\pi = py - wx_1 - wx_2 = p[\ln(x_1 + 1) + \ln(x_2 + 1)] - w_1 x_1 - w_2 x_2 \tag{1}$$

Maximization of profits means setting the derivatives of π (with respect to each input, x_1 and x_2) equal to 0. From equation (1), here that means

$$\frac{p}{x_1 + 1} - w_1 = 0 \tag{2}$$

$$\frac{p}{x_2 + 1} - w_2 = 0 \tag{3}$$

Without any further substitution, equations (2) and (3) can each be re–arranged to get the unconditional demand of the firm for each input :

$$x_1 + 1 = \frac{p}{w_1} \tag{4}$$

$$x_2 + 1 = \frac{p}{w_2} \tag{5}$$

From the definition of the production function ($y = \ln(x_1 + 1) + \ln(x_2 + 1)$), equations (4) and (5) imply that the firm's profit maximizing output is

$$y = \ln\left(\frac{p}{w_1}\right) + \ln\left(\frac{p}{w_2}\right) \tag{6}$$

or

$$y = 2\ln p - \ln w_1 - \ln w_2 \tag{7}$$

(so that the firm's choice of output increases with the price of the output, and decreases with the price of each input).

*Q*3. What is the (long–run total) cost function for a firm with a production function

$$y = x_1 + \ln(x_2 + 1) \qquad ?$$

*A*3. Given that $f(x_1, x_2) = x_1 + \ln(x_2 + 1)$, here

$$MP_1 = 1 \tag{8}$$

$$MP_2 = \frac{1}{x_2 + 1} \tag{9}$$

The condition for cost minimization is always that the TRS equal the ratio of the input prices. Here

$$TRS = \frac{MP_1}{MP_2} = x_2 + 1$$

so that cost minimization requires

$$x_2 + 1 = \frac{w_1}{w_2} \tag{10}$$

or

$$x_2 = \frac{w_1}{w_2} - 1 \tag{11}$$

Equation (11) is the firm's conditional demand function for input #2.

(Actually, equation (11) makes sense only if $w_1 > w_2$; otherwise it defines a negative demand for input 2. If $w_2 > w_1$, then the firm should actually use only input #1, and should set $x_2 = 0$.)

Since $y = x_1 + \ln(x_2 + 1)$, therefore

$$x_1 = y - \ln(x_2 + 1) \tag{12}$$

if the firm is to meet its target level of output. Substituting from (11),

$$x_1 = y - \ln w_1 + \ln w_2 \tag{13}$$

is the firm's conditional demand function for input #1 (at least when $w_1 > w_2$). Since the firm's long– total cost

$$C(w_1, w_2, y) = w_1 x_1 + w_2 x_2$$

equations (11) and (13) imply that

$$C(w_1, w_2, y) = w_1[y - \ln w_1 + \ln w_2] + w_2[\frac{w_1}{w_2} - 1] \tag{14}$$

or

$$C(w_1, w_2, y) = w_1 y - w_1 \ln w_1 + w_1 \ln w_2 + w_1 - w_2 \tag{15}$$

$Q4$. If a firm's short–run total cost function is

$$SRTC = \frac{y^2}{z} + z$$

when it produces an output level of y using a plant size z :

(i) For a given plant size z, what output level minimizes the short–run average cost?

(ii) If the plant size z can be varied in the long–run, what is the equation of the firm's long–run average cost curve?

$A4$. First of all, given that

$$SRTC = \frac{y^2}{z} + z$$

it follows that the short–run average cost is

$$SRAC = \frac{SRTC}{y} = \frac{y}{z} + \frac{z}{y} \tag{16}$$

To see the shape of the short–run average cost curve, as the output level y varies, note that the derivative of (16) is

$$SRAC'(y) = \frac{1}{z} - \frac{z}{y^2} \tag{17}$$

At very small levels of output (near $y = 0$) the negative term in (17) is very large, so that the average cost curve slopes down. But as y increases, this negative term shrinks, so that eventually the positive term in (17) dominates, and the SRAC curve starts to slope up.

The bottom of this average cost curve is the level of output y for which $SRAC'(y) = 0$. From equation (17), this cost–minimizing output level y^* is the value of y for which

$$\frac{1}{z} = \frac{z}{y^2} \tag{18}$$

or

$$\frac{y}{z} = \frac{z}{y}$$

which means that $y^* = z$.

This answers part i of the question : if the plant size is z, then the output which minimizes the cost per unit of output is a level of output $y^* = z$ here.

To answer part ii, find out what is the firm's optimal plant size z, if it chooses to produce the output level y. That is, in the long run, the firm will choose its cost–minimizing level of plant size z. Since the total cost of y units of output is $\frac{y^2}{z} + z$ if the firm has a plant size of z, the best plant size for an output level y is the plant size for which $\frac{y^2}{z} + z$ is smallest. Minimizing this cost with respect to z means finding a plan size z for whch

$$\frac{\partial SRTC}{\partial z} = -\frac{y^2}{z^2} + 1 = 0 \tag{19}$$

Here equation (17) says that setting $z = y$ ensures that $y^2/z^2 = 1$ so that $SRTC$ is at a minimum.

Therefore, the best plant size, in the long run, for an output level of y, is $z = y$. Call this best plant size $\hat{z}(y)$. Since the firm can vary plant size to minimize costs in the long run, it follows that

$$LRTC = \frac{y^2}{\hat{z}(y)} + \hat{z}(y) \tag{20}$$

Since $\hat{z}(y) = y$, therefore

$$LRTC = 2y \tag{21}$$

From equation (21), the firm's long–run average cost curve is horizontal, $LRAC = 2$, although each of the short–run average cost curves is U–shaped. In this example, the $LRAC$ curve actually is tangent to each $SRAC$ curve at the bottom of that $SRAC$ curve, because here there are constant returns to scale in the long run.

$Q5$. What is the equation of a firm's short–run supply curve, if its short–run total cost function is

$$SRTC = \frac{y^2}{z} + z$$

when it produces an output level of y using a plant size z?

$A5$. The short–run supply curve is the firm's marginal cost curve, where that marginal cost exceeds the short–run average variable cost. From differentiation of $SRTC = \frac{y^2}{z} + z$,

$$SRMC = 2\frac{y}{z} \tag{22}$$

What is the firm's variable cost here, in the short run? It's the total cost, minus the fixed cost. The total cost is $\frac{y^2}{z} + z$. The fixed cost is the cost of producing a zero level of output, $SRTC(0)$, which here equals z. So

$$SRTVC = SRTC - FC = \frac{y^2}{z} + z - z = \frac{y^2}{z} \tag{23}$$

So the short–run average variable cost is

$$SRAVC = \frac{SRTVC}{y} = \frac{y}{z} \tag{24}$$

Comparison of equations (22) and (24) shows that the marginal cost here always exceeds the average variable cost. Therefore the firm's short–run supply curve here is its **entire** marginal cost curve.

To express the quantity supplied as a function of price, since (22) implies that $2\frac{y}{z} = p$ when the firm sets $p = MC$, so that the equation of the firm's short–run supply function is

$$y = \frac{z}{2}p \tag{25}$$