AS/ECON 4070 3.0AF Answers to Assignment 1 October 2006

$Q1$. What are **all** the allocations on the contract curve for the following 2–person 2–good exchange economy?

Person 1's preferences can be represented by the utility function

$$u(x_1, y_1) = \sqrt{x_1} + 2\sqrt{y_1}$$

Person 2's preferences can be represented by the utility function

$$U(x_2, y_2) = 2\sqrt{x_2} + \sqrt{y_2}$$

The economy's total endowment of good X is 120, and the economy's total endowment of good Y is 60.

$A1$. In an exchange economy, an allocation will be Pareto optimal if, for any two goods, all people have the same marginal rate of substitution (MRS) between those goods.

Here there are only 2 goods, and only 2 people. Each person's MRS is the ratio of the marginal utilities of the 2 goods. Since $u(x_1, y_1) = \sqrt{x_1} + 2\sqrt{y_1}$, for person 1

$$\frac{\partial u}{\partial x_1} = \frac{1}{2\sqrt{x_1}}$$

$$\frac{\partial u}{\partial y_1} = \frac{1}{\sqrt{y_1}}$$

implying that her MRS is

$$MRS_1 \equiv MU_x^1 / MU_y^1 = \frac{1}{2\sqrt{x_1}} / \frac{1}{\sqrt{y_1}} = \frac{1}{2}\sqrt{\frac{y_1}{x_1}}$$

Analogously, for person 2,

$$\frac{\partial U}{\partial x_2} = \frac{1}{\sqrt{x_2}}$$

$$\frac{\partial U}{\partial y_2} = \frac{1}{2\sqrt{y_2}}$$

so that

$$MRS_2 = 2\sqrt{\frac{y_2}{x_2}}$$

For some allocation (x_1, y_1, x_2, y_2), the two people's indifference curves will be tangent in the Edgeworth box if and only if their MRS's are equal, or

$$\frac{1}{2}\sqrt{\frac{y_1}{x_1}} = 2\sqrt{\frac{y_2}{x_2}} \qquad (1-1)$$

Now there are 120 units of good X, so that $x_2 = 120 - x_1$ if the good is divided between the two people. There are 60 units of good Y, so that $y_2 = 60 - y_1$. Therefore, equation $(1-1)$ can be written

$$\frac{1}{2}\sqrt{\frac{y_1}{x_1}} = 2\sqrt{\frac{60-y_1}{120-x_1}} \qquad (1-2)$$

If both sides of equation (1-2) are squared, it becomes

$$\frac{1}{4}\frac{y_1}{x_1} = 4\frac{60-y_1}{120-x_1} \qquad (1-3)$$

which can be written

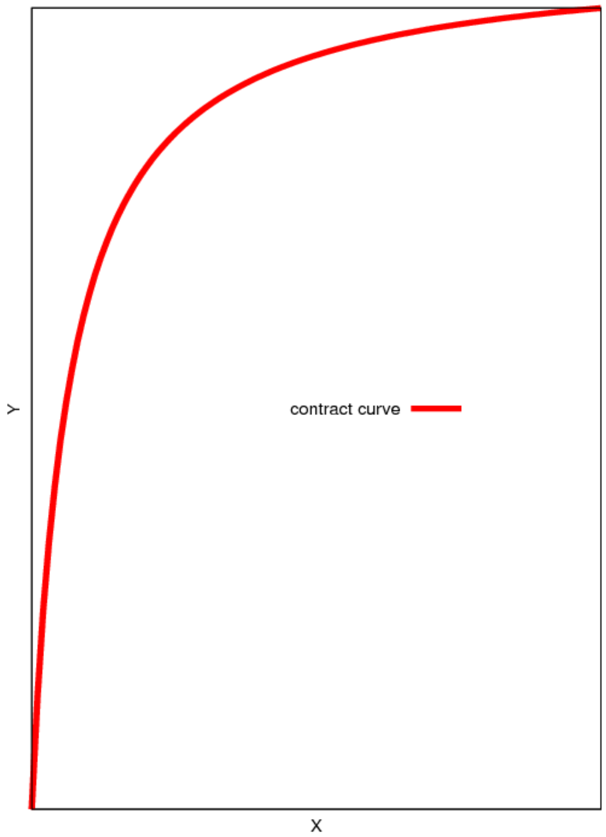
$$y_1(120-x_1) = 16(x_1(60-y_1) \qquad (1-4)$$

Equation $(1-4)$ defines a curve, relating y_1 to x_1. Any pair (x_1, y_1) satisfying this equation (with x_1 and y_1 non–negative, and with $x_1 \leq 120$, $y_1 \leq 60$) will be a Pareto optimal allocation.

This equation can also be written

$$y_1 = \frac{64x_1}{8+x_1} \qquad (1-5)$$

which defines an upward–sloping contract curve in the Edgeworth box, starting at $x_1 = 0, y_1 = 0$, going through points such as $x_1 = 10, y_1 = 35.55$, $x_1 = 30, y_1 = 50.52$, and winding up at $x_1 = 120, y_1 = 60$, at the top right corner of the Edgeworth box.



$Q2$. Suppose that a country contains N people, each of whom supplies (inelastically) the same amount of labour. There are many firms in the country, each of which has the same production function

$$y = 2\sqrt{nk}$$

where y is the quantity of cloth produced by the firm, n is the number of workers hired by the firm, and k is the amount of cotton used by the firm.

Firms sell cloth at a price of \$1 per metre ; this price is determined on world markets, and is not influenced by the country (which is very small).

The market for labour in the country is perfectly competitive, since there are many firms, and many workers.

All cotton is imported. The world price of cotton, r per tonne, is not influenced by the country (which is small).

The country's government can levy a tax t per tonne on firms' employment of cotton.

How does the wage earned by workers vary with the tax on cotton?

$A2$. The price paid by firms for cotton will be $r + t$, if the government imposes a tax of t per tonne, on top of the world price of r. Let w denote the wage earned by workers.

Since firms are perfectly competitive, to maximize profits each firm uses an amount of cotton k such that

$$p(MP_K) = r + t$$

where p is the world price of the cloth that they produce (and where MP_K is the marginal product of cotton). Similarly, each firm wants to employ a quantity n of labour such that

$$p(MP_N) = w$$

Given the production function $y = 2\sqrt{nk}$,

$$MP_K = \frac{\partial y}{\partial k} = \sqrt{\frac{n}{k}} \tag{2 – 1}$$

$$MP_L = \frac{\partial y}{\partial n} = \sqrt{\frac{k}{n}} \tag{2 – 2}$$

That means that profit maximization by firms leads them to hire quantities of labour and cotton such that

$$p\sqrt{\frac{n}{k}} = r + t \tag{2 – 3}$$

$$p\sqrt{\frac{k}{n}} = w \tag{2 – 4}$$

Dividing $(2 - 4)$ by $(2 - 3)$,

$$\frac{k}{n} = \frac{w}{r + t} \tag{2 – 5}$$

Now plug $(2-5)$ back into $(2-4)$:

$$w = p\sqrt{\frac{w}{r+t}} \qquad (2-6)$$

or

$$w^2 = p^2 \frac{w}{r+t} \qquad (2-7)$$

so that

$$w = \frac{1}{r+t} \qquad (2-8)$$

when the world price p of cloth equals 1.

Here, taxing cotton more heavily reduces the wage rate, by driving up the cost of cloth production.

*Q*3. In the country described in question #2 above, suppose that the country distributes all its revenue from taxation of foreign cotton equally to each worker. (If the tax is negative – that is, if cotton use is subsidized — then the cost of the subsidy is spread equally over all workers.)

What cotton tax is best for the workers?

*A*3. If the tax rate is t per tonne of cotton, (from the answer to question #2) each worker will receive a wage of $1/(r+t)$: raising the tax rate lowers her wage rate. But she also receives her share of the tax revenue, which may increase with the tax rate.

How much tax revenue is collected? That depends on the total quantity K of cotton which is used by firms in the country.

Equation $(2-5)$ says that each firm uses $w/(r+t)$ tonnes of cotton per worker. Since each firm uses the same ratio of cotton to workers k/n, in the country the overall ratio of cotton used K, to workers used N is the same

$$\frac{K}{N} = \frac{w}{r+t} \qquad (3-1)$$

From equation $(2-8)$ then,

$$K = \frac{1}{(r+t)^2}N \qquad (3-2)$$

is the total quantity of cotton used in the country, when there are N workers, and when the tax is t per tonne of cotton.

Equation $(3-2)$ means that each worker in the country would collect

$$\frac{tK}{N} = \frac{t}{(r+t)^2} \qquad (3-3)$$

dollars from the cotton tax, if the revenue were distributed equally to all workers. Her overall income is her wage income w, plus her share of the tax revenue. If i denotes her overall income, then

$$i = w + \frac{tK}{N} = \frac{1}{r+t} + \frac{t}{(r+t)^2} = \frac{r+2t}{(r+t)^2} \qquad (3-4)$$

Differentiating,
$$\frac{\partial i}{\partial t} = -\frac{2t}{(r+t)^3} \qquad (3-5)$$
Equation $(3-5)$ shows that the overall income per worker is a **decreasing** function of the tax rate, whenever the tax rate is positive. It also is an **increasing** function of t whenever $t < 0$. The best tax here is no tax at all. The revenue collected by increasing the tax is more than offset by the reduction in the wage rate cuased by the reduced demand for cotton. On the other hand, it is also not a good idea to subsidize cotton use : the wage increase induced by a subsidy on cotton use is less than the cost of the subsidy.

*Q*4. What is the incidence of a 100% tax (calculated as a percentage of the net [before–tax] price) on haircuts, if the market for haircuts is perfectly competitive, if the supply curve in the market has the equation
$$Q^s = (p_s)^3$$
and the demand curve has the equation
$$Q^D = \frac{1}{(P_D)^2}$$
where p_s is the price received by sellers, P^D is the price paid by buyers, Q^s is the quantity supplied, and Q^D is the quantity demanded?

*A*4. In this question, both the demand curve and the supply curve have constant elasticities.
$$\epsilon_s = \frac{\partial Q^s}{\partial p_s}\frac{p_s}{Q^s} = 3(p_s)^2\frac{p_s}{(p_s)^3} = 3 \qquad (4-1)$$
$$\epsilon_D = -\frac{\partial Q^D}{\partial P_D}\frac{P_D}{Q^D} = 2(P_D)^{-3}\frac{P_D}{(P_D)^{-2}} = 2 \qquad (4-2)$$
so that the simple elasticity approximation formula says that buyers should bear a fraction $\epsilon_s/(\epsilon_s + \epsilon_D)$, or 60% of a tax.

How good is that approximation?

Here the actual equilibrium price can be calculated. Let τ be the tax rate, as a percentage of the net price. Then
$$P_D = p_s(1+\tau) \qquad (4-3)$$
so that, the equilibrium condition $Q^s = Q^D$ implies that
$$(p_s)^3 = \frac{1}{(p_s)^2(1+\tau)^2} \qquad (4-4)$$
or
$$(p_s)^5 = \frac{1}{(1+\tau)^2} \qquad (4-5)$$

When there is no tax, $\tau = 0$, so that $p_s = 1$ solves equation $(4-5)$. A 100% tax on the net–of–tax price means that $\tau = 1$, so that equation $(4-5)$ becomes

$$p_s = [\frac{1}{(1+1)^2}]^{1/5} = \frac{1}{4}^{1/5} \qquad (4-6)$$

This means that, with the 100% tax, p_s is about 0.75786, so that $P_D = 1.51572$, and the tax collects revenue of 0.7586. So buyers bear a share $(1.51572 - 1)/(0.75786)$ of the tax. That's about 68% of the burden born by buyers, and 32% by sellers. So the very simple approximation understates the share of the burden born by buyers, but not by that much.

Another approximation used is that the share of the tax born by the buyers is

$$\frac{\epsilon_s}{\epsilon_s + (p_s/P^D)\epsilon_D}$$

Here p_s/P_D equals 1 in the "before" equilibrium, and it equals 1/2 in the "after" equilibrium, when there is a 100% tax. So this formula gives a share of the burden for buyers of 60% if the "before" value of p_s/P_D is used, a share of $3/(3+(1/2)2$, or 75% if the "after" value is used. Taking an average for p_s/PD, or 3/4 (the averrage of 1 and 1/2, gives an answer of $3/(3+(3/4)2$, or 66.7%, which is a pretty good approximation for the share of the burden born by buyers.

*Q*5. What would be the incidence of a $1 tax on a good, if the good was sold by a Bertrand duopoly, if the cost of production of both firms in the duopoly were

$$TC(q) = 12q$$

(where $TC(q)$ is the total cost of producing q units of the good), and if aggregate demand for the good obeyed the equation

$$Q^d = 800 - 50P^D$$

(where Q^d is the quantity demanded, and P^D is the [tax–included] price paid by consumers)?

[Recall : In a Bertrand duopoly, two firms each choose their price simultaneously, and non–cooperatively. Buyers buy from the lower–priced firm ; if firms charge the same price, they each get half the aggregate demand.]

*A*5. The important property of Bertrand oligopoly for this question is that the equilibrium price must equal the firms' marginal cost, if they set prices simultaneously, and if they each have the same constant marginal cost.

This property is derived in Varian's [chapter 27] or Nicholson's [chapter 22] intermediate micro textbooks. The logic behind the result? Suppose P_i is the price (including taxes) chosen by firm i. If $P_1 < P_2$, then all the customers would buy from firm 1 ; this situation could not be an equilibrium (if prices were above firms' costs), since firm 2 would want to lower its prices, in order to get some customers. Similarly, it is not an equilibrium if $P_1 > P_2$, if both prices are above cost

; now firm 1 would want to lower its price. If both firms charged the same price, then they would split the market. But then firm 1 (for example) could double its sales, simply by lowering its price a tiny amount, and undercutting firm 2. If the price were above cost, then this strategy (lower the price a little and steal all the market) would increase the firm's profits, so there cannot be an equilibrium in which both firms charge the same price, unless the price equals cost. Each firm's incentive to undercut the other leads to an equilibrium in which price has been driven down to cost, and in which each firm makes zero economic profits, just as it would if there were many firms in a perfectly competitive market.

In this question, firms do have constant marginal costs. Since $TC = 12q$, then $MC = 12$, in the absence of any taxes. So if there were no taxes, each firm would charge a price of 12, the 2 firms would split the total market demand of $800 - 50(12) = 200$, and each firm would make zero economic profits. If the statutory incidence of the tax were on firms, then each firm's total cost would go from $12q$ to $(12 + t)q$, if t is the unit tax on the good. Now each firm's marginal cost is $12 + t$. So, in Bertrand competition, each firm would charge a price of $12 + t$. The demand price would go from 12 to $12 + t$. Buyers would bear 100% of the tax, just as they would if the good were sold in a perfectly competitive market.

As usual, the statutory incidence does not matter here. If buyers were legally required to pay the unit tax, then each firm's marginal cost would stay unchanged, at 12. In Bertrand competition, then, each firm would charge a price of $p_s = 12$ in equilibrium (if the statutory incidence were on buyers). With $t = 1$, the tax would raise the buyers' tax–included price P_D from 12 to 13, and drive market demand down from 200 to $800 - 50(13) = 150$. Again buyers bear 100% of the tax, and sellers 0%, since sellers are making 0 economic profits in equilibrium.