AS/ECON 4070 3.0AF   Answers to Assignment 1   October 2007

*Q*1. Find all the efficient allocations in the following 2–person, 2–good, 2–input economy.

The 2 goods, food and clothing, are produced using labour and machinery as input. There is a fixed quantity of 150 person-hours of labour available, which can be allocated between food and clothing production. There is also a fixed quantity of 130 machine–hours of machinery available, which can be allocated between food and clothing production.

The total quantity X of food produced depends on how much labour and machinery are used in the food industry :

$$X = \frac{1}{2}(L_X + K_X)$$

where L_X is the number of person–hours of labour used in food production, and K_X is the number of machine–hours of machinery used in food production.

The total quantity Y of clothing produced is

$$Y = \sqrt{L_Y K_Y}$$

Person 1's preferences can be represented by the utility function

$$u_1 = \ln x_1 + \ln y_1$$

where x_1 and y_1 are person 1's quantities consumed of food and of clothing respectively. Person 2's preferences can be represented by the utility function

$$u_2 = \ln x_2 + \ln y_2$$

*A*1 There are three sets of conditions for Pareto optimality : exchange efficiency, production efficiency and overall efficiency.

Here it is best to begin with efficiency in production. The condition for efficiency in production is that the marginal rate of transformation (the MRTS) between the 2 inputs be the same in both industries. The MRTS is the ratio of the marginal products of the two inputs.

In food production, since $X = K_X + L_X$, the marginal product of labour is 1/2, and so is the marginal product of machinery. The MRTS equals 1, regardless of what are the quantities used of labour and machinery. In this case, the isoquants between labour and capital in food production are straight lines (with slope of -1).

In clothing production, differentiation of the production function $Y = \sqrt{L_Y K_Y}$ with respect to L_Y and K_Y shows that

$$MP_L^Y = \frac{1}{2}\sqrt{\frac{K_Y}{L_Y}} \qquad (1-1)$$

$$MP_K^Y = \frac{1}{2}\sqrt{\frac{L_Y}{K_Y}} \qquad (1-2)$$

so that the MRTS in the clothing industry is

$$MRTS^Y = \frac{MP_L^Y}{MP_K^Y} = \frac{K_Y}{L_Y} \tag{1 - 3}$$

Since $MRTS^X = 1$, regardless of how much labour and machinery are used in food production, efficiency in production, which requires $MRTS^Y$ to equal $MRTS^X$, requires that

$$\frac{L_Y}{K_Y} = 1 \tag{1 - 4}$$

Any allocation of inputs (L_X, K_X, L_Y, K_Y) to industries, will be efficient if (i) L_X, K_X, L_Y and K_Y are all non–negative, (ii) $L_X + L_Y = 200$ and $K_X + K_Y = 100$, and (iii) equation $(1 - 4)$ holds.

Since equation $(1 - 4)$ can be written $K_Y = L_Y$, an allocation of inputs to industries will be efficient if we assign Z units of labour and of machinery to the clothing industry — where $0 < Z < 130$ —, and the remaining $150 - Z$ units of labour and $130 - Z$ units of machinery to the food industry.

Such an allocation of labour and machinery is efficient in production, and yields the following outputs : the clothing industry produces

$$Y = \sqrt{K_Y L_Y} = \sqrt{Z \cdot Z} = Z$$

and the food industry produces

$$X = \frac{1}{2}[(150 - Z) + (130 - Z)] = 140 - Z$$

So the production possibility frontier is a straight line, with a slope of -1, if we graph X along the horizontal axis and Y along the vertical : each increase in Z will increase Y by 1 and decrease X by 1.

The line starts at the point $(140, 0)$, which is the output combination if all the labour and all the machinery is used in food production. It moves up and to the left, with a slope of -1, all the way up to the point $(10, 130)$, which is the output combination resulting if all the machinery is used in clothing production.

Can we produce more than 130 units of clothing? Yes, but now the only input left to transfer from the food industry is labour : when $Z = 130$, all the machinery is already used in clothing production, so the only way to increase Y is by shifting labour. If all 130 units of machinery are allocated to clothing production, and $Z > 130$ units of labour are also allocated there, then

$$Y = \sqrt{130Z}$$

$$X = \frac{1}{2}[150 - Z]$$

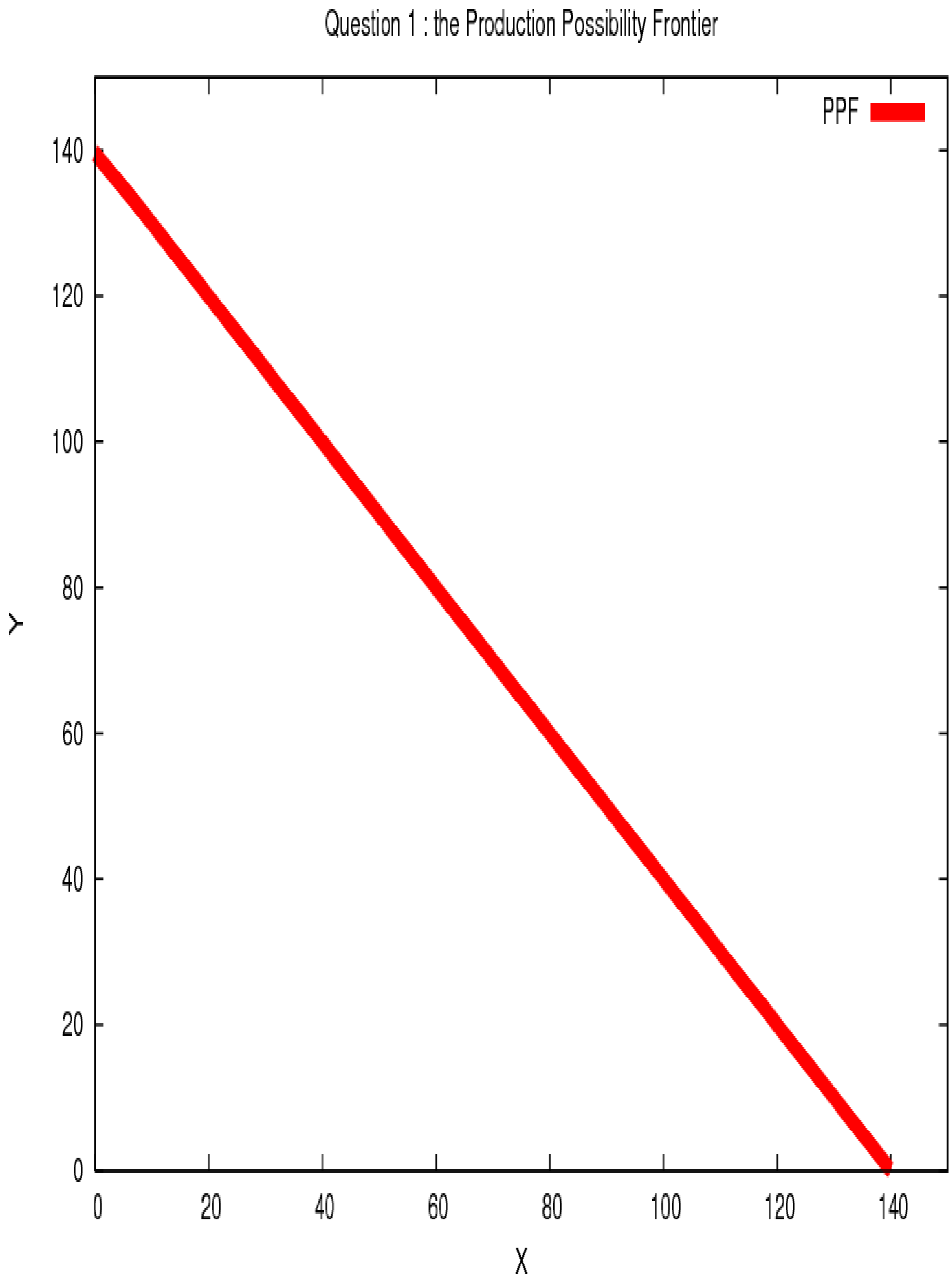

Figure 1 : the production possibility frontier is (mostly) a straight line

so that, above $(10, 130)$ the PPF is curved. Each unit of labour shifted from food to clothing production lowers X by $1/2$, and raises Y by $\frac{\partial Y}{partial L_Y} = \frac{1}{2}\sqrt{\frac{K_Y}{L_Y}} = \frac{1}{2}\sqrt{\frac{130}{L_Y}}$.

Figure 1 depicts the PPF.

A Pareto optimal allocation must be efficient in exchange. Efficiency in exchange requires that each person's marginal rate of substitution between the two goods (food and clothing) be the same. The MRS is the ratio of the marginal utilities of food and clothing consumption.

Since person 1's preferences are represented by the utility function $u_X = \ln x_1 + \ln y_1$, her marginal utilities are

$$MU_X^1 = \frac{\partial u_1}{\partial x_1} = \frac{1}{x_1} \tag{1-5}$$

$$MU_Y^1 = \frac{\partial u_1}{\partial y_1} = \frac{1}{y_1} \tag{1-6}$$

Her MRS is just the ratio of these marginal utilities, the rate at which she wishes to substitute one good for the other,

$$MRS^1 = \frac{MU_Y^1}{MU_X^1} = \frac{x_1}{y_1} \tag{1-7}$$

Similarly, for person 2,

$$MRS^2 = \frac{x_2}{y_2} \tag{1-8}$$

Since efficiency in exchange requires that each person's MRS be the same, efficiency in exchange requires

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} \tag{1-9}$$

In this economy, the two people must consume the two goods in the same proportion.

So suppose that the economy produces X units of food and Y units of clothing. Condition $(1-9)$ says that person 1 should get the same share of food as she gets of clothing. If s denotes person 1's share of the food, an allocation is efficient in exchange if person 1 gets sX units of food and sY units of clothing, and person 2 gets the remaining $(1-s)X$ units of food and $(1-s)Y$ units of clothing.

The final efficiency condition is overall efficiency, that each person's MRS equal the overall MRT. The MRT is just the (absolute value of the) slope of the MRT. Above, it was shown that the MRT was 1, at least when clothing production was no greater than 130.

Since each person consumes goods in the same proportion [if the allocation is efficient in exchange], then $x_1/y_1 = x_2/y_2 = X/Y$. So an allocation will be overall efficient if

$$MRS^1 = MRS^2 = \frac{X}{Y} = MRT = 1 \tag{1-10}$$

So equation $(1-10)$ requires that $X = Y$ for overall efficiency. We can produce those quantities if we allocate Z units of machinery and of labour to clothing production, so that

$$Y = Z = X = 140 - Z \tag{1-11}$$

The solution to equation $(1-11)$ is $Z = 70$.

Summarizing, in this economy, and allocation is efficient only if we allocate 70 units of labour, and 70 units of machinery to the clothing industry, with the remaining 80 units of labour and 60 units of machinery going to the food production, resulting in 70 units of food and 70 units of clothing being produced.

The quantities of output ($X = 70$ and $Y = 70$) can then be divided between the two people. If person 1 gets a share s of the clothing, then she should also get the same share s of the food, if the allocation is efficient.

$Q2$. An imaginary economy has 2 million people. Each person's preferences in this economy can be represented by the utility function

$$u = xy$$

where u is the person's utility, x her consumption of wheat, and y her consumption of barley.

There is only 1 input to production in this economy, land. A hectare of land can be used to grow 1 tonne of wheat, or to grow 1 tonne of barley.

One million of the people in the economy are poor, and own 1 hectare of land each. The other one million each own 5 hectares of land. (So that there are 6 million hectares of land in total, available for wheat or barley growing.)

Small competitive firms rent land from the people, and use the land to grow wheat and barley, which they sell on competitive markets. [There is no home production : the only way that people can get wheat or barley is to buy it from firms.]

The government taxes wheat sales. They confiscate half the wheat sold in any transaction. So if a firm sells 2 tonnes of wheat to a person, they must also give 2 tonnes of wheat to the government as well.

The government takes all the wheat that it collects, and divides it equally among the poor people.

What is the competitive equilibrium allocation in this economy?

$A2$. In this economy, the MRT must be 1. There is only 1 input (land), and transferring one land from barley production to wheat production lowers barley output by 1 tonne, and raises wheat output by 1 tonne.

Let p be the price of wheat received by wheat farmers here. Then the price of barley must also be p : the price ratio (of the prices received by producers must equal the MRT, which is 1).

Here the price of land must also be p per hectare : each hectare of land produces 1 tonne of barley, so that the value of the marginal product of 1 hectare must be p, and this value of the marginal product must equal 1 in a competitive economy.

Note that the price p is the price of wheat received by farmers. That means that the price of wheat paid by consumers is $2p$, since the government is taxing wheat at 100% (as a proportion of the pre–tax price).

A consumer, with an income of m to spend on wheat and barley, wants to find her most preferred bundle (w, b) of wheat and barley, subject to her budget constraint that $(2p)w + pb = m$ — since $2p$ is the price she pays per tonne of wheat, and p is the price she pays per tonne of barley.

So she finds a bundle where her marginal rate of substitution equals the price ratio. Since her preferences are represented by the utility function $u = xy$,

$$MU_w = b$$

$$MU_b = w$$

and

$$MRS_{wb} = \frac{MU_w}{MU_b} = \frac{b}{w} \qquad (2-1)$$

Setting the MRS equal tot he price ratio, she chooses a bundle for which

$$\frac{b}{w} = \frac{2p}{p} = 2 \qquad (2-2)$$

since $2p$ is the (tax–inclusive) price of wheat and p is the price of barley.

Equation $(2-2)$ says that this consumer wants to consume twice as much barley as wheat. But since all consumers have the same utility functions, and all pay the same prices, everyone wants to consume twice as much barley as wheat.

The whole economy, then, is nothing but people who want to consume twice as much barley as wheat. In equilibrium (since quantity supplied must equal quantity demanded), the economy produces twice as much barley as wheat.

That means that the economy produces 4 million tonnes of barley, and 2 million tonnes of wheat in equilibrium, since there are 6 million hectares of land available, and each hectare of land can yield 1 tonne of wheat or barley.

So how much tax does the government collect? The price received by wheat sellers is p, and the government collects the same amount in tax. It collects $2p$ million dollars in tax revenue. This revenue is divided among the 1 million poor person.

So each poor person receives $2p$ from the tax revenue. She also owns 1 hectare of land, and receives p from renting out the land. So her income is $3p$. Given that she wants to consume twice as much barley as wheat, she consumes 0.75 tonnes of wheat, and 1.5 tonnes of barley (which costs exactly $2p(0.75) + 1.5p = 3p$, which is what she has to spend).

Each rich person has an income of $5p$. Again the rich person wants to consume twice as much barley as wheat, so buys 1.25 tonnes of wheat and 2.5 tonnes of barley (which costs exactly $2p(1.25) + p(2.5) = 5$).

So the equilibrium allocation has each poor person consuming the bundle $(0.75, 1.5)$ and each rich person consuming the bundle $(1.25, 2.5)$, where I have listed wheat consumption first.

*Q*3. Is the equilibrium allocation in question #2 above Pareto optimal?

Explain precisely why or why not.

*A*3. Two (or more) ways of answering here.

i Here, each person's MRS between wheat and barley is 2, and the economy's MRT between wheat and barley is 1. Therefore, the allocation is not overall efficient, and thus not Pareto optimal.

ii More directly, there are allocations which are Pareto preferred. The equilibrium allocation gives each poor person a utility of $(0.75)(1.5) = 1.125$ and each rich person a utility of $(1.25)(2.5) = 3.125$.

The allocation in which each poor person gets 1.1 tonnes of wheat and 1.1 tonnes of barley, and in which each rich person gets 1.9 tonnes of wheat and 1.9 tonnes of barley is feasible : this allocation requires $(1.1+1.1+1.9+1.9)$ million hectares of land, which is exactly what is available.

This new allocation gives each poor person a utility of $(1.1)(1.1) = 1.21$ and each rich person a utility of $(1.9)(1.9) = 3.61$ So both types of people are better off than in the equilibrium allocation : the new allocation is Pareto–preferred to the equilibrium allocation.

*Q*4. What is the incidence of a 100% tax (calculated as a percentage of the net [before–tax] price) on haircuts, if the market for haircuts is perfectly competitive, if the supply curve in the market has the equation

$$Q^s = \sqrt{\frac{p_s}{2}}$$

and the demand curve has the equation

$$Q^D = \frac{16}{P_D}$$

where p_s is the price received by sellers, P^D is the price paid by buyers, Q^s is the quantity supplied, and Q^D is the quantity demanded?

*A*4. Again, there are a couple ways of answering the question.

i Solve explicitly for the prices, before and after the tax has been imposed.

Let τ be the tax rate, expressed as a fraction of the seller's price p_s (so that $\tau = 0$ when there is no tax, and $\tau = 1$ when there is a 100 percent tax). Then

$$P_D = (1+\tau)p_s$$

Setting quantity demanded equal to quantity supplied,

$$\sqrt{\frac{p_s}{2}} = \frac{16}{(1+\tau)p_s} \qquad (4-1)$$

Taking the squares of both sides of $(4-1)$,

$$\frac{p_s}{2} = \frac{256}{(1+\tau)^2 p_s^2} \qquad (4-2)$$

so that

$$p_s^3 = \frac{512}{(1+\tau)^2} \qquad (4-3)$$

or

$$p_s = \frac{8}{(1+\tau)^{2/3}} \qquad (4-4)$$

(since $8^3 = 512$). So when there is no tax, $\tau = 0$ and

$$p_s = P_D = 8$$

A 100 percent tax makes $\tau = 1$ and $(1+\tau)^{2/3}) \approx 1.587$, so that

$$p_s \approx 5.04 \quad P_D \approx 10.08$$

when there is a 100 percent tax.

The price paid by buyers has gone up by $10.08 - 8 = 2.08$ and the price received by sellers has gone down by $8 - 5.05 = 2.95$, so that buyers bear approximately 40 percent of the tax.

ii Elasticities give a good approximation of the tax incidence. In this case the elasticity of demand is

$$\eta^D = -\frac{\partial Q^D}{\partial P_D}\frac{P_D}{Q^D} = \frac{16}{(P_D)^2}P_D\frac{P_D}{16} = 1$$

and the elasticity of supply is

$$\eta_s = \frac{\partial Q_s}{\partial p_s}\frac{p_s}{Q_s} = \frac{1}{4}(p_s)^-1/2p_s\frac{2}{(p_s)^{1/2}} = \frac{1}{2}$$

This formula predicts that buyers should bear a share

$$\frac{\eta_s}{\eta_s + \eta_D}$$

of the tax if the tax is small, and this share equals 1/3.

iii A more accurate elasticity formula can be applied if the tax is not so small. In this case the share of the tax born by the buyers is

$$\frac{(1+\tau)\eta_s}{(1+\tau)\eta_s + \eta_D}$$

Which tax rate τ to use here? The tax initially was 0. Then it was 100%. If we take the average of those two values, then $\tau = 0.5$. Since $\eta_s = 1/2$ and $\eta_D = 1$ (and both are constant in this case), using this average implies that buyers should bear a share

$$\frac{(1.5)(0.5)}{(1.5)(0.5)+1} = \frac{3}{7}$$

of the tax, which is very close to the answer obtained in part *i*.

$Q5$. What would be the incidence of a \$6 unit tax in the following market?

The market demand curve for the good has the equation

$$Q^D = 15 - P^D$$

where P^D is the price paid by buyers, and Q^D the quantity demanded.

There is a single firm which can produce as much or as little of the good as it wants, at a constant cost of \$5 per unit.

There are many other firms. But each other firm can only produce the good at a cost of \$8 per unit. (These other firms also produce under constant returns to scale.)

The single low–cost firm sets its price to maximize its profit, knowing that it will not sell any of the good if it charges a higher price than the other firms.

[You may assume that the customers all buy from the low–cost firm if the low–cost firm charges exactly the same price as the high–cost firms.]

$A5$. The single firm would like to maximize its profits by charging the monopoly price. But it has to worry about the other, high–cost firms.

There are many other firms, each with a cost of $8 + t$ (where $t = 0$ initially, and then $t = 6$). So the monopoly must charge a price of 8 or less initially, and 14 or less when the tax is levied : if it charges a higher price than the high–priced firms' costs, then high–priced firms will enter the industry, and undercut the monopoly.

Initially, the monopoly's profit will be $Q^D(P^D - 5)$, if it charges a price of 8 or less, since its costs are \$5 per unit. Since $Q^D = 15 - P^D$, its profit (when there is no tax) is

$$\pi^0 = (15 - P^D)(P^D - 5) \qquad (5-1)$$

Choosing a price P^D to maximize profits, means setting the derivative of expression $(5-1)$ with respect to P^D equal to 0, or

$$(15 - P^D) - (P^D - 5) = 0 \qquad (5-2)$$

or

$$P^D = 10$$

But a price of 10 is too high! The firm will be undercut by high–cost competitors, who can make a profit if they can charge a price of 8 or more.

So, in the absence of a tax, the (potential) monopoly's best strategy is to charge a price of 8. That price keeps out competition. Any higher price would lead to entry by competitors. Any lower price will lower profit. (You can check that the profit defined by equation $(5-1)$ is a declining function of P^D when $P^D < 10$.)

With a tax of \$6, the monopoly's cost rises to 11. Its profit becomes

$$\pi^1 = (15 - P^D)(P^D - 11) \qquad (5-3)$$

Maximizing expression $(5-3)$ with respect to P^D means setting its derivative equal to 0, or

$$(15 - P^D) - (P^D - 11) = 0 \tag{5-4}$$

or

$$P^D = 13$$

But the monopoly still should worry about potential entry. The high–cost competitors now have unit costs of 14, since the tax of 6 is added on to their production costs of 8 per unit. The monopoly is safe from competition if it charges its profit–maximizing price of 13, since that is greater than its rivals costs.

So the introduction of the tax of $6 raises the monopoly's price from 8 to 13. Buyers bear 5/6 of the tax.

Normally, an "unthreatened" monopoly, with constant marginal costs, and a linear demand curve, will pass on only half of a tax to its buyers. But here, because of the threat of entry by other firms, the monopoly actually passes on more than half the tax. That is because the tax raises the rivals' price more than it raises the monopoly's profit–maximizing price. (It raises the rivals' costs by $6, and the monopoly's profit–maximizing price [if there were no threat of competition] by $3.) Here the tax makes the monopoly less threatened by the competition, and causes it to pass on more of the tax to buyers than an unthreatened monopoly would.