AS/ECON 4070 3.0AF   Answers to Assignment 1   October 2008

*Q*1. Find the equation of the production possibility curve in the following 2–good, 2–input economy.

Food and clothing are both produced using labour and machinery as inputs. The quantity produced of food is

$$X = \sqrt{L_X K_X}$$

where X is the quantity produced of food, L_X is the quantity of labour used in food production, and K_X is the quantity of machinery used in food production. The quantity produced of clothing is

$$Y = L_Y + 4K_Y$$

where Y is the quantity produced of clothing, L_Y is the quantity of labour used in clothing production and K_Y is the quantity of machinery used in clothing production.

The economy has a fixed total endowment of 200 units of labour and 90 units of machinery.

*A*1. A pair of quantities (X, Y) is on the production possibility curve if and only if the allocation of inputs to the two industries is efficient in production. Efficiency in production requires that the marginal rates of technical substitution — the ratios of the marginal products of the two inputs — are equal in the two industries.

Given that $X = \sqrt{L_X K_X}$, the marginal products of labour and machinery in the food industry are

$$MP_L^X \equiv \frac{\partial X}{\partial L_X} = \frac{1}{2}\sqrt{\frac{K_X}{L_X}} \qquad (1-1)$$

$$MP_K^X \equiv \frac{\partial X}{\partial K_X} = \frac{1}{2}\sqrt{\frac{L_X}{K_X}} \qquad (1-2)$$

so that

$$MRTS^X \equiv \frac{MP_L^X}{MP_K^X} = \frac{K_X}{L_X} \qquad (1-3)$$

In the clothing industry, where $Y = L_Y + 4K_Y$,

$$MP_L^Y \equiv \frac{\partial Y}{\partial L_Y} = 1 \qquad (1-4)$$

$$MP_K^Y \equiv \frac{\partial Y}{\partial K_Y} = 4 \qquad (1-5)$$

so that

$$MRTS^Y \equiv \frac{MP_L^Y}{MP_K^Y} = \frac{1}{4} \qquad (1-6)$$

Efficiency in production requires that $MRTS^X = MRTS^Y$, which from equations $(1-3)$ and $(1-6)$ means that

$$\frac{K_X}{L_X} = \frac{1}{4}$$

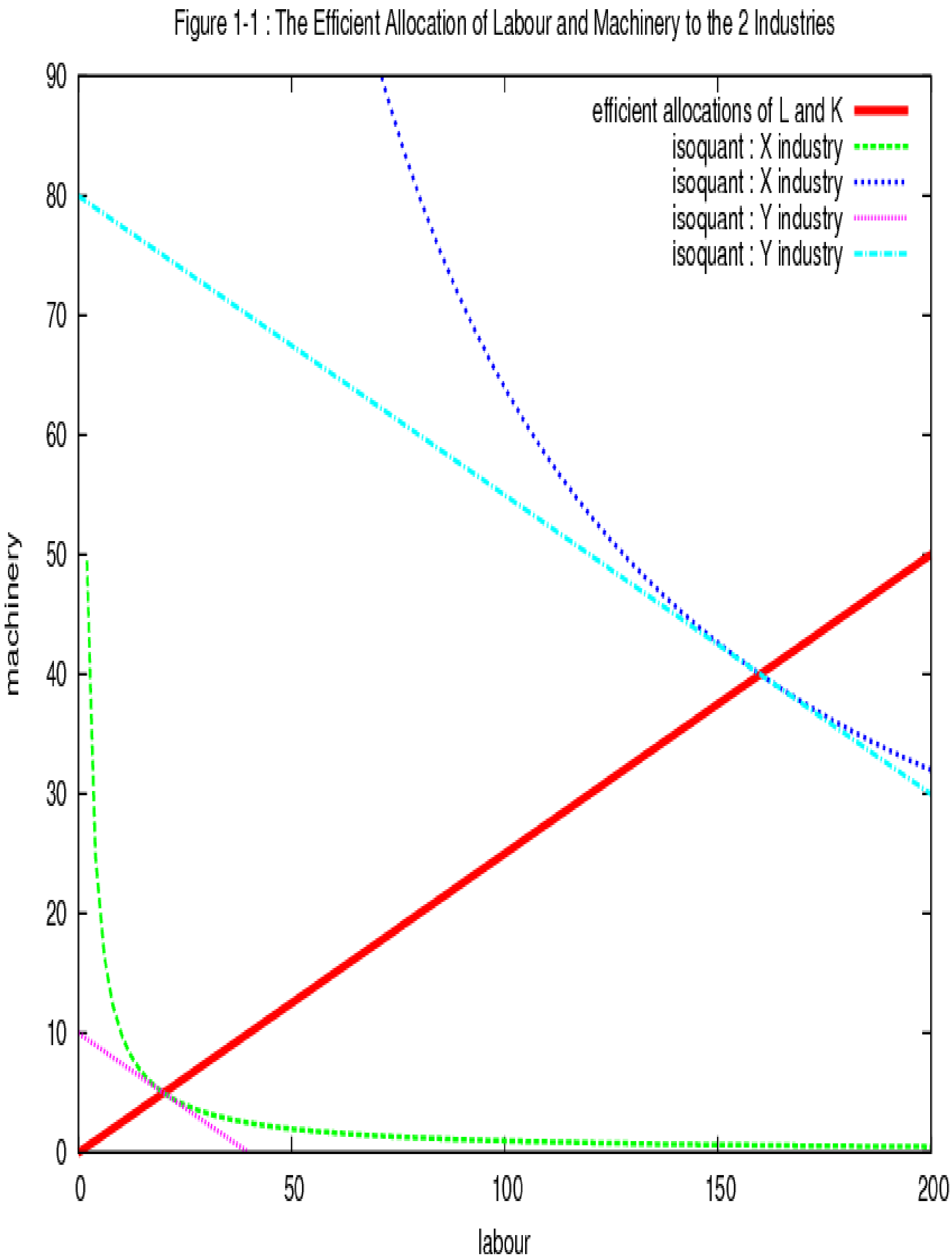

Figure 1–1 : the contract curve - efficient input allocations

or

$$L_X = 4K_X \tag{1-7}$$

Figure $1-1$ shows the efficient allocation of labour and machinery in an Edgeworth Box. In it, labour and machinery allocated to the X industry are measured from the bottom left, and labour and machinery allocated to the Y industry are measured from the top right. The efficient allocations of the inputs are those for which the isoquants for the two industries are tangent, those satisfying equation $(1-7)$. [In the figure, the red "contract curve" hits the right edge of the box at $L_X = 200, K_X = 50, L_Y = 0, K_Y = 40$. At that point, all labour is allocated to the X industry. The "contract curve" actually continues up the right edge of the box, to the top right–hand corner. Allocations for which $L_X = 200, 50 < K_X \leq 90, L_Y = 0, K_Y = 90 - K_X$ are efficient : here labour is relatively more productive in the food industry, but it is impossible to move any more labour from the clothing to the food industry, since all available labour is being used in food production.]

So suppose that L_X units of labour and K_X units of machinery are used in the food industry. If the allocation is efficient, then $(1-7)$ implies that $L_X = 4K_X$ so that

$$X = \sqrt{K_X L_X} = \sqrt{(4K_X)(K_X)} = 2K_X \tag{1-8}$$

From $(1-7)$ and $(1-8)$, if the food industry produces X units of food, and if the production plan is efficient, then

$$K_X = \frac{X}{2} \tag{1-9}$$

$$L_X = 4K_X = 2X \tag{1-10}$$

Since there are 200 units of labour, and 90 units of machinery, then

$$L_Y = 200 - L_X$$

$$K_Y = 90 - K_X$$

Since the output of clothing, $Y = L_Y + 4K_Y$, therefore $(1-9)$ and $(1-10)$ imply that

$$Y = 200 - 2X + 4(90 - \frac{X}{2}) = 560 - 4X \tag{1-11}$$

Equation $(1-11)$ is the equation of the production possibility curve. It shows how much clothing can be produced, if the economy is efficient in production, and if X units of food are being produced. In this case, it is a straight line, with a slope of -4.

[Equation $(1-11)$ is actually only valid when food production X is 100 or less. At $X = 100$, equation $1-10$) implies that all the labour available in the economy is being used in food production. To produce more than 100 units of food, we cannot add any more labour (there isn't any more). So we have to add machinery. So if $X > 100$, then $L_X = 200$, and

$$X = \sqrt{200K_X}$$

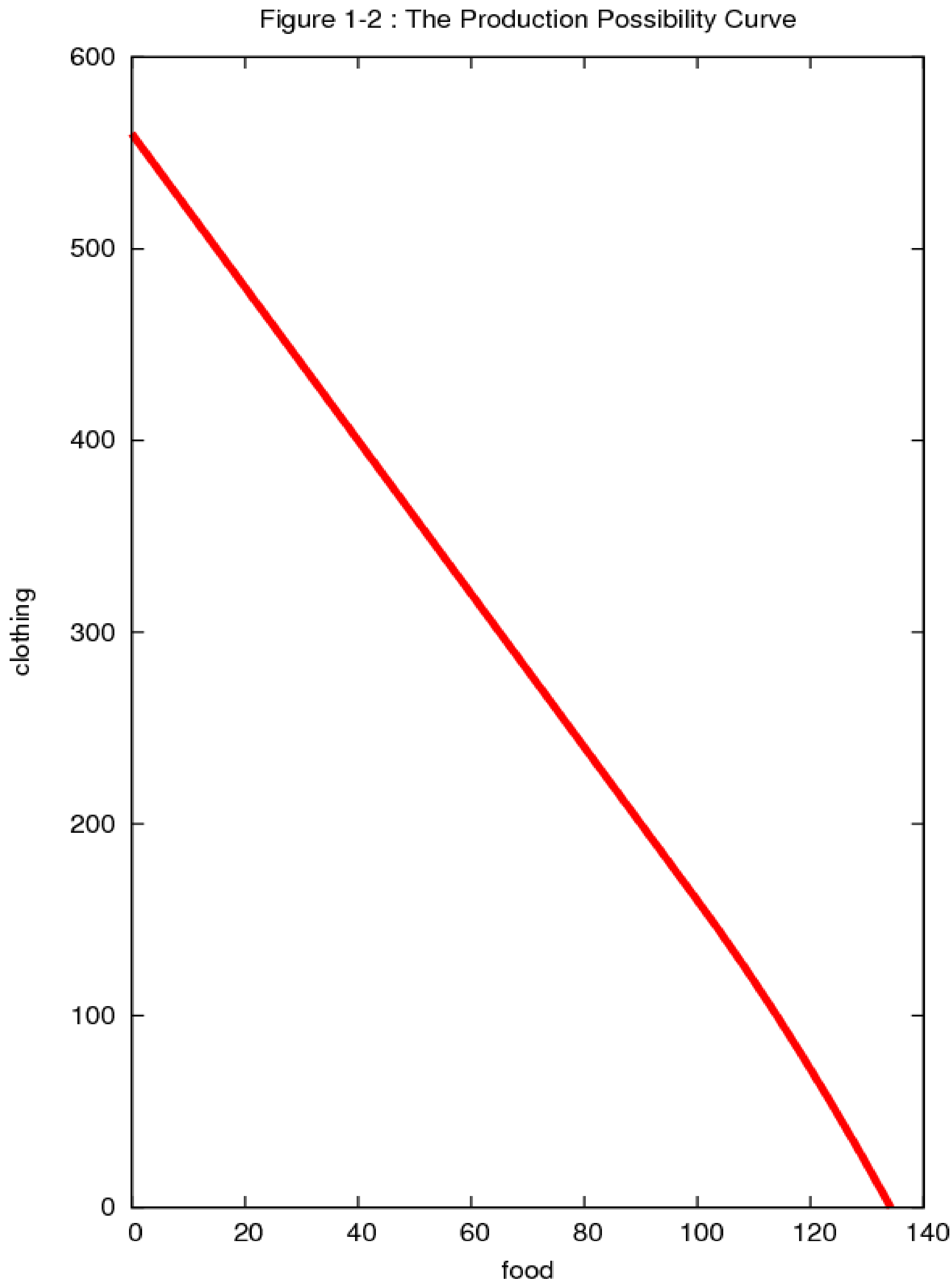

Figure 1–2 : the production possibility curve

so that

$$K_X = \frac{X^2}{200} \tag{1-12}$$

In this case, with all the labour being used in food production, clothing is produce from machinery alone. Equation $(1-12)$ implies that there are $K_Y = 90 - \frac{X^2}{200}$ units of machinery available for clothing production when food production is $X > 100$. That means

$$Y = 4K_Y = 360 - \frac{4X^2}{200} \tag{1-13}$$

So $(1-11)$ is the equation of the production possibility curve if $X \leq 100$, and $(1-13)$ is the equation of the part of the possibility curve for which $100 < X < 134.16$ (at $X = 60\sqrt{5} \approx 134.16$, all the labour and all the machinery is used in food production, so that $Y = 0$)]

Figure $1-2$ illustrates the production possibility curve. [Note that it's a straight line from $(0, 560)$ to $(100, 160)$, but then is actually slightly curved below $(100, 160)$, since all the labour has been allocated to clothing production when $X = 100$.]

*Q*2. In the economy described in question #1, there are 290 people. There are 90 capitalists, each of whom owns one unit of machinery, and no labour. There are 200 workers, each of whom owns one unit of labour, and no machinery. Each of the 290 people has the same preferences, represented by the utility function

$$u(x, y) = xy$$

where x is the person's consumption of food, and y is her consumption of clothing.

Show that, if all firms and people behave as perfect competitors, there is an equilibrium in which the price of clothing is \$1, the price of food is \$4, the wage of each worker is \$1, and the return to each unit of machinery is \$4.

Is this equilibrium Pareto optimal? Explain briefly.

*A*2. Since only relative prices matter, one of the prices can be chosen arbitrarily : that is, any good can be picked as the numéraire, the good in terms of which all values are measured.

So we can make clothing the numéraire, which means setting $p_Y = 1$.

Under perfect competition, firms hire inputs up to the point where the value of their marginal products equal the return they earn. In the clothing industry, that means it must be true that $p_Y MP_L^Y = w_L$ and $p_Y MP_K^Y = w_K$, where w_L and w_K are the returns to labour and machinery respectively. If $p_Y = 1$, equations $(1-4)$ and $(1-5)$ imply that we must have

$$p_Y MP_L^Y = 1 = w_L \tag{2-1}$$

$$p_Y MP_K^Y = 4 = w_K \tag{2-2}$$

in any competitive equilibrium, if $p_Y = 1$.

The slope of the production possibility curve represents the opportunity cost of one good in terms of the other. In particular, equation $(1-11)$ implies that increasing food production by 1 unit must lead to a reduction of 4 units in clothing production. That is, the opportunity cost of each unit of food [if $X < 100$] is 4 units of clothing. So if $p_Y = 1$, it must be the case that $p_X = 4$.

Firms in the food industry must hire labour and machinery up to the point where the value of their marginal products equal the return they earn. From $(1-1)$ and $(1-2)$, if $p_X = 4$ then

$$p_X MP_L^X = 4\frac{1}{2}\sqrt{\frac{K_X}{L_X}} = 1 \qquad (2-3)$$

$$p_X MP_K^X = 4\frac{1}{2}\sqrt{\frac{L_X}{L_X}} = 4 \qquad (2-4)$$

Equations $(2-3)$ and $(2-4)$ can hold only if $L_X = 4K_X$, as is required for production efficiency.

On the consumption side, each person, whether worker or capitalist, wants to choose a consumption bundle for which her marginal rate of substitution (MRS) equals the price ratio. If preferences are represented by the utility function $u(x, y) = xy$, then

$$MRS \equiv \frac{MU_x}{MU_y} = \frac{y}{x} \qquad (2-5)$$

Since $p_X = 4$ and $p_Y = 1$, each person wants a consumption bundle where her MRS, y/x equals the price ratio, 4. That means that for each person, whether worker or capitalist, $y = 4x$.

So for the whole economy, aggregate clothing production Y must be 4 times aggregate food production X. Since we are efficient in production, the production plan (X, Y) must satisfy equation $(1-11)$. We must have $Y = 560 - 4X$ and $Y = 4X$. Therefore $4X = 560 - 4X$, or

$$X = 70 \qquad (2-6)$$

If food production X is 70, then equation $(1-11)$ implies that $Y = 560 - 4(70) = 280$.

So the production plan for the economy is $(L_X, K_X) = (140, 35)$, and $(L_Y, K_Y) = (60, 55)$, with $X = \sqrt{(140)(35)} = 70$ and $Y = 60 + 4(55) = 280$.

Each worker earns $w_L = 1$. She chooses a consumption bundle with 4 times as much clothing as food. Since the price of food is exactly 4 times the price of clothing, that means that she spends the same amount on food as she does on clothing : $p_X x = p_Y y$. So she spends half her earnings on each good, and has a consumption bundle of

$$(x_1, y_1) = (\frac{1}{8}, \frac{1}{2}) \qquad (2-6)$$

Capitalists consume food and clothing in the same ratio as workers. So they too spend half of their income on food and half on clothing. Since their income is higher, 4, their consumption bundle is

$$(x_2, y_2) = (\frac{1}{2}, 2) \qquad (2-7)$$

Note that $200x_1 + 90x_2 = 70 = X$, and $200y_1 + 90y_2 = 280 = Y$.

That completes the description of the economy's equilibrium allocation. Is it efficient? The first fundamental theorem of welfare economics says that it must be efficient. But this can be checked. Here

$$MRTS^X = \frac{1}{4} = MRTS^Y$$

$$MRS_1 = \frac{y_1}{x_1} = 4 = \frac{y_2}{x_2} = MRS_2$$

And

$$MRT = 4 = MRS$$

so that all the conditions for Pareto optimality are satisfied at this allocation.

*Q*3. In the model described in questions 1 and 2, suppose now that the government imposes a high tax on the use of machinery in the clothing industry : clothing firms must pay a tax of \$3 to the government for every \$1 they spend on the use of machinery (that is, there is a tax of 300% imposed on the use of machinery in the clothing industry).

The revenue from this tax is distributed only to machinery owners, and it is distributed equally to them, so that each capitalist gets a fraction 1/90 of the tax revenue.

Show that, if all firms and people behave as perfect competitors, there is an equilibrium in which the price of clothing is \$1, the price of food is \$2, the wage of each worker is \$1, and the return to each unit of machinery is \$1. Is this equilibrium Pareto optimal? Explain briefly.

*A*3. Again, only relative prices matter, so that all prices can be expressed relative to the price of any good. So we can again choose clothing as the numéraire by setting $p_Y = 1$.

If w_K denotes the return per machine earned by the owners of the machines, then firms in the clothing industry must pay $4w_K$ for each machine they use : the return to the owner of the machine, plus a tax equal to three times that return. Profit maximization by clothing firms means hiring inputs up to the point at which the value of their marginal product equals the unit cost — including any taxes — of the input. So, given $(1-4)$ and $(1-5)$, in equilibrium

$$p_Y MP_L^Y = 1 = w_L \qquad (3-1)$$

$$p_Y MP_K^Y = 4 = 4w_K \qquad (3-2)$$

so that in equilibrium $w_L = w_K = 1$ when $p_Y = 1$.

Firms in the food industry do not have to pay any taxes on their use of machinery. So their profit maximization implies (given equations $(1-1)$ and $(1-2)$) that

$$p_X MP_L^X = \frac{p_X}{2}\sqrt{\frac{K_X}{L_X}} = w_L = 1 \qquad (3-3)$$

$$p_X MP_K^X = \frac{p_X}{2}\sqrt{\frac{L_X}{K_X}} = w_K = 1 \qquad (3-4)$$

Equations $(3-3)$ and $(3-4)$ imply that

$$K_X = L_X \tag{3-5}$$

That means that each unit of food is produced using 1 unit of labour and 1 unit of machinery : equation $(3-5)$ implies equal quantities of each input must be used, and the fact that $X = \sqrt{K_X L_X}$ implies that it must be exactly one unit each of labour and capital that are being used.

So the cost of producing 1 unit of food is 1 : each unit of food requires 1 unit each of labour and capital, and the cost of each of those units is 1. In competitive equilibrium, this price of food must equal the unit cost of producing 1 unit of food. (Food is produced in perfect competition, under constant returns to scale, so that the price of food must equal its marginal cost.) Therefore, in equilibrium, it must be the case that

$$p_X = 2$$

On the consumption side, each person's MRS equals $\frac{y}{x}$, and each person sets that MRS equal to the price ratio $\frac{p_X}{p_Y} = 2$. So $y_i = 2x_i$ for each person, be she worker or capitalist.

In aggregate, total clothing production Y must be twice the total food production X.

$$Y = 2X \tag{3-6}$$

Since $K_X = L_X$, therefore

$$X = \sqrt{K_X L_X} = L_X \tag{3-7}$$

Total clothing production Y equals $L_Y + 4K_Y$. Since $L_Y = 200 - L_X$, $K_Y = 90 - K_X$ and $L_X = K_X = X$, therefore

$$Y = 200 - X + 4(90 - X) = 560 - 5X \tag{3-8}$$

Combining equations $(3-6)$ and $(3-8)$,

$$560 - 5X = 2X$$

or

$$X = 80 \tag{3-9}$$

which implies that

$$Y = 2X = 160 \tag{3-10}$$

Equations $(3-9)$ and $(3-10)$ show that the outcome is not Pareto efficient. The output combination $X = 90$, $Y = 200$ is on the production possibility curve defined in $(1-11)$: it is possible to produce more food and more clothing than the outcome $X = 80$, $Y = 160$ defined by $(3-9)$ and $(3-10)$.

Since $X = 80$, therefore $K_X = L_X = 80$, so that $L_Y = 120$ and $K_Y = 10$. The revenue raised by the tax on the use of machinery in the clothing industry is 30 : \$3 per unit on each of the 10 units of machinery used in the clothing industry.

So the income of each worker is 1. And the income of each capitalist is now the return to capital, plus the capitalist's share of the tax revenue. That share is $30/90 = 1/3$, so that each capitalist has income of 4/3.

Each person, worker or capitalist, still spends half her income on food and half on clothing : since $y_i/x_i = p_X/p_Y$ therefore $p_X x_i = p_Y y_i$. So each worker now spends half her income of 1 on food, and half her income on clothing, yielding an allocation of

$$(x_1, y_1) = (\frac{1}{4}, \frac{1}{2}) \tag{3-11}$$

(Recall that the price of food is 2, so spending half of her income of \$1 on food means consuming 1/4 units of food.) Capitalists spend half of their income on food, and half on clothing, so that their allocation is

$$(x_2, y_2) = (\frac{1}{3}, \frac{2}{3}) \tag{3-12}$$

Again $200x_1 + 90x_2 = 80 = X$ and $200y_1 + 90y_2 = 160 = Y$.

Even though capitalists get all the tax revenue, the tax on the use of machinery in the clothing industry has made capitalists worse off, and has made workers better off.

But — as already mentioned — the allocation is not Pareto efficient. With lump–sum transfers, and with no tax on the use of machinery in the clothing industry, it would be possible to make both workers and capital owners better off than they are in this equilibrium.

*Q*4. What would be the approximate incidence of a unit tax of \$2 levied on sellers in a perfectly competitive market in which the quantity of the good demanded by buyers was

$$Q^D = 60 - (P^D)^2$$

and the quantity supplied by sellers was

$$Q^s = 10p_s - 36$$

where Q^D is the quantity demanded by buyers, Q^s the quantity supplied by sellers, P^D the price paid by buyers and p_s the price received by sellers?

*A*4. If the unit tax is t, then

$$P^D = p_s + t \tag{4-1}$$

The market is in equilibrium if quantity demanded equals quantity supplied,

$$60 - (P^D)^2 = 10p_s - 36 \tag{4-2}$$

Substituting from $(4-1)$ for P^D in $(4-2)$,

$$60-(p_s+t)^2 = 10p_s - 36 \tag{4 – 3}$$

or

$$(p^s)^2 + (10+2t)p_s(t^2-96) = 0 \tag{4 – 4}$$

Equation $(4-4)$ is a quadratic equation, which can be solved for p_s ; it can be written in the form $(p_s)^2 + bp_s - c$, where $b \equiv 10+2t$ and $c = t^2 - 96$. The quadratic formula says that the solution to this equation is

$$p_s = -\frac{b}{2} + \frac{1}{2}\sqrt{b^2-4c} = -5 - t + \frac{1}{2}\sqrt{484+40t} \tag{4 – 5}$$

(There is also a second solution to this quadratic, but the second solution gives a negative value for p_s, which makes no sense.) When there is no tax, so that $t=0$, equation $(4-5)$ becomes

$$p_s = -5 + \frac{1}{2}\sqrt{484} = 6 \tag{4 – 6}$$

With a tax of 2 per unit, $(4-5)$ becomes

$$p_s = -7 + \frac{1}{2}\sqrt{564} \approx 4.874 \tag{4 – 7}$$

So a tax of \$2 reduces the price received by sellers from \$6 to approximately \$4.87. That means that sellers are losing $6-4.87$ from the tax, and bearing a share $(6-4.87)/2 = 0.565$ of the burden of the \$2 tax. Since the price paid by buyers rises from \$6 to $4.874+2 = 6.874$ because of the tax, buyers bear a share $(6.87-6)/2 = 0.435$ of the tax.

So sellers bear about 56 or 57 percent of the burden of the tax, and buyers bear the remaining 43 or 44 percent.

Of course, an approximate answer for the burden of the tax can be obtained from using the slopes of the supply and demand functions, or from using the elasticities. The problem is, the slope of the demand curve is not a constant ; it depends on the value of the price. (That's true for the elasticity of demand as well.) So we have to know something about what the price is before we can plug in the formula.

Specifically, since $D(P^D) = 60-(P^D)^2$ and $S(p_s) = 10p_s - 36$, are the equations of the demand and supply functions, we have

$$D'(P^D) = -2P^D \tag{4 – 8}$$

$$S'(p_s) = 10 \tag{4 – 9}$$

so that demanders will bear a share of the tax of

$$\frac{S'}{S'-D'} = \frac{10}{10+2P^D} \tag{4 – 10}$$

To evaluate $(4-10)$ we need to know the demand price P^D. If we already knew the initial demand price was 6, then $(4-10)$ says that demanders should bear a fraction $10/(10+2(6)) \approx 0.444$ of the tax burden, which is not a bad approximation to the true result. If we somehow knew the price after the tax was imposed, $p^D \approx 6.874$, then $(4-10)$ says that demanders bear a fraction $10/(10+2(6.874)) \approx 0.421$ of the tax burden, which is also a pretty close approximation of the true result. (Of course if we know the initial price $P^D = 6$ and the final demand price with the tax $P^D \approx 6.874$, then we don't need to use the formula.)

But even without knowing the exact price, $(4-10)$ gives us some idea of the share born by demanders. If the quantity supplied is non–negative, it must be true that $10p_s - 36 > 0$ so that $p_s > 3.6$. Since the demand price P^D must be at least p_s in value, $(4-10)$ says that the share of the tax born by demanders cannot exceed $10/(10+2(3.6)) \approx 0.581$, so demanders' share of the burden cannot exceed about 58 percent. If the quantity demanded is non–negative, it must be true that $(P^D)^2 \leq 60$, so that $P^D \leq \sqrt{60} \approx 7.75$, which means that the share of the tax born by demanders must be at least $10/(10+2(7.75)) \approx 0.392$.

So without even solving for the equilibrium prices (with or without taxes) the approximation formulae implies a range for the share of the tax born by demanders : this share must be somewhere between 39 percent and 58 percent.

*Q*5. Suppose that a good is produced by two firms, each of which has a constant marginal cost of production of \$6 per unit.

Consumers regard the two firms' products as perfect substitutes for each other, and will always buy from the cheaper firm (if the firms were to charge different prices from each other).

The aggregate demand for the product has the equation

$$Q^D = 240 - 30P^D$$

where Q^D is the quantity demanded by buyers, and P^D the price paid by buyers.

The two firms in this oligopolistic market behave as *Bertrand* duopolists. That is, firms each set their own prices, taking as given the price charged by the other firm (and anticipating how buyers will respond to their pricing decisions).

What will be the incidence of a \$1 unit tax levied on both sellers in this market?

*A*5. To answer this question, we have to know what happens in Bertrand competition. Under Bertrand competition, each firm sets its own price, taking the other firm's price as given. So if firm 1 were to charge a higher price that firm 2, it would not get any customers ; buyers would all buy from the other firm. That gives each firm a strong incentive to undercut the other firm, since the higher–priced firm gets no sales.

Even when the two firms' prices are equal, the incentive to undercut remains. If firm 1 and firm 2 charged the same price, and split the market, then firm 1 would still want to undercut :

cutting its price a little would make it the (only) low–priced firm, and would gain it the whole market.

So this undercutting drives the price down. In equilibrium, both firms must charge a price equal to their marginal cost.

[This result may be familiar from Econ 2350 : it's derived in section 27.9 of Varian's intermediate microeconomics text, for example.]

What is the marginal cost here? With no tax, it's \$6. With a unit tax of \$1, it's \$7. So the tax raises the equilibrium price from \$6 to \$7 — just as it would if the industry were perfectly competitive. With price competition, if there are two (or more) firms with the same constant marginal costs [and if buyers regard all firms' products as identical] then buyers bear 100% of an excise tax.