AS/ECON 4070 3.0AF   Answers to Assignment 2   November 2006

$Q1$. If a person's preferences can be represented by a utility function

$$u(X, Y, Z) = Z + 20 \ln XY$$

where X, Y and Z are the person's quantities consumed of food, clothing and other goods, and if the net–of–tax price of each of the goods is 1, and if the person's income is 60, what would be the total excess burden of a tax of \$1 on good X and a tax of \$3 on good Y?

$A1$. The first step is to find out how much the person demands of the different goods. Her marginal utility of consumption of the three goods, given the utility function, are

$$MU_X = \frac{\partial u}{\partial X} = \frac{20}{X}$$

$$MU_Y = \frac{\partial u}{\partial Y} = \frac{20}{Y}$$

$$MU_Z = \frac{\partial u}{\partial Z} = 1$$

She chooses a consumption bundle (X, Y, Z) so that her marginal rate of substitution between any two goods equals the ratio of the prices of the goods : that is, the slope of her indifference curve between any 2 goods must equal the ratio of the goods' prices.

So

$$MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{Y}{X} = \frac{P_X}{P_Y} \quad (1-1)$$

$$MRS_{XZ} = \frac{MU_X}{MU_Z} = \frac{20}{X} = \frac{P_X}{P_Z} \quad (1-2)$$

$$MRS_{YZ} = \frac{MU_Y}{MU_Z} = \frac{20}{Y} = \frac{P_Y}{P_Z} \quad (1-3)$$

where (P_X, P_Y, P_Z) are the (tax-inclusive) prices she pays for the 3 goods.

Equations $(1-2)$ and $(1-3)$ give the demand functions for goods X and Y respectively :

$$X^D = 20\frac{P_Z}{P_X} \quad (1-4)$$

$$Y^D = 20\frac{P_Z}{P_Y} \quad (1-5)$$

To find the demand for good Z, use the person's budget constraint $P_X X + P_Y Y + P_Z Z = M$, where M is her income ; this equation implies that

$$Z^D = \frac{M - P_X X^D - P_Y Y^D}{P_Z}$$

so that $(1-4)$ and $(1-5)$ imply that

$$Z^D = \frac{M}{P_Z} - 40 \qquad (1-6)$$

Note that in this case (because the person has **quasi–linear** preferences), her quantities demanded of goods X and Y do not vary with her income M. That implies (from the Slutsky equation) that her compensated demand functions for these two goods are the same as her uncompensated demand functions, and that the compensating variation to any tax on goods X and Y will be the same as the equivalent variation.

Initially, with no tax, $P_X = P_Y = P_Z = 1$, and $M = 60$, so that

$$X^D = 20; Y^D = 20; Z^D = 20$$

The tax raises P_X to 2 and P_Y to 4, so that after the tax

$$X^D = 10; Y^D = 5; Z^D = 20$$

How much would we have to compensate her for the taxes, so that she would stay on the same indifference curve as she was initially, with $X = Y = Z = 20$? Initially, her utility was

$$20 + 20 \ln 400 \qquad (1-7)$$

If $P_X = 2$, $P_Y = 4$, $P_Z = 1$, and her income was $M + CV$, then equation $(1-6)$ shows that she would consume $Z^D = 20 + CV$, and equations $(1-4)$ and $(1-5)$ show that $X = 10$ and $Y = 5$ (since her demands for these 2 goods do not vary with her income). Therefore, the utility she would get is

$$20 + CV + 20 \ln 50 \qquad (1-8)$$

Comparing $(1-7)$ and $(1-8)$, the amount of compensation she would need to receive, in order to undo the damage of the taxes, is

$$CV = 20[\ln 400 - \ln 50] = 20 \ln 8 \qquad (1-9)$$

Checking a table of natural logarithms, $20 \ln 8 = 41.5888$.

How much revenue does the tax raise?

$$Rev = t_X X^D + t^Y Y^D = (1)(10) + (3)(5) = 25$$

So the excess burden of the tax is $CV - Rev$, or (approximately) 16.5888.

In this example, the quantity demanded of good X does not depend on the price of good Y, and vice versa. That means that the area under the demand curves for the 2 taxed goods will give an exact measure of the excess burden.

The excess burden in the market for X is the area between the heights of 1 and 2, under the demand curve for good X, minus the tax revenue collected. So (from the equation $(1-4)$ of the demand curve for X)

$$EB_X = \int_1^2 \frac{20}{P_X} dP_X - (1)(10) = 20(\ln 2 - \ln 1) - 10 = 20 \ln 2 - 10 \qquad (1-10)$$

Similarly

$$EB_Y = \int_1^4 \frac{20}{P_Y} dP_Y - (3)(5) = 20(\ln 4 - \ln 1) - 15 = 20 \ln 4 - 15 \qquad (1-11)$$

Since $\ln 8 = \ln 4 + \ln 2$, equations $(1-10)$ and $(1-11)$ show that $EB_X + EB_Y$ equals the overall expression for EB (approximately 16.5888) derived in expression $(1-9)$.

*Q*2. A person chooses how much to work when young, how much to consume when young, and how much to consume when old.

If she works H hours per week when young, and spends C_Y on consumption when young, then she can save $wH - C_Y$ per week. The interest rate on her saving is r (so that she gets to consume $(1+r)S$ per week when old, if she saves S per week when young). Her wage rate is w per hour.

Her preferences can be represented by the utility function

$$u(C_Y, C_O, H) = 20 \ln C_Y + 20 \ln C_O - H$$

where C_O is her consumption expenditure per week when old.

The government needs to raise a certain amount of revenue R, from sales taxation. It can tax consumption in each period. Since the government can borrow or lend at the interest rate r, it is the present value of tax revenue which matters ; the present value of tax collections

$$T_Y + \frac{T_O}{1+r}$$

must equal R, where T_Y and T_O are the sales tax revenue collected in the two periods of the person's life.

What is the relation between the sales tax rate when the person is young, and the sales tax rate when she is old, if the government wants to raise the required present value of revenue at minimum harm to the person?

*A*2. The first, step, again, is to find out her labour supply, and her consumption demand, as a function of the prices she faces. The person's budget constraint is

$$C_O(1 + \tau_O) = (1 + r)(wH - (1 + \tau_Y)C_Y) \qquad (2-1)$$

where τ_Y and τ_Y are the tax rates on consumption in the two periods. Equation $(2-1)$ can be re–arranged into

$$H = \frac{1+\tau_Y}{w} C_Y + \frac{1+\tau_O}{w(1+r)} C_O \qquad (2-2)$$

Substituting from $(2-2)$ into the person's utility function, she will be choosing C_Y and C_O so as to maximize

$$20 \ln C_Y + 20 \ln C_O - \frac{1+\tau_Y}{w} C_Y - \frac{1+\tau_O}{w(1+r)} C_O \qquad (2-3)$$

Picking C_Y and C_O to maximize $(2-3)$ means setting the derivatives of $(2-3)$ with respect to C_Y and C_O equal to 0. So

$$\frac{20}{C_Y} = \frac{1+\tau_Y}{w}$$

$$\frac{20}{C_O} = \frac{1+\tau_O}{w(1+r)}$$

or

$$C_Y = 20 \frac{w}{1+\tau_Y} \qquad (2-4)$$

$$C_O = 20 \frac{w(1+r)}{1+\tau_O} \qquad (2-5)$$

Equations $(2-4)$ and $(2-5)$ show that consumption demand in any one period is independent of the tax rate on consumption in the other period. They also show that the elasticity of consumption demand with respect to the "price" of consumption in each period [$(1+\tau_Y)$ in the first period, and $(1+\tau_O)/(1+r)$ in the second period] is the same : 1. Therefore, the Ramsey rule shows that it is optimal to tax consumption in the two periods at the same rate.

Equations $(2-4)$ and $(2-5)$ imply that

$$\frac{C_O}{C_Y} = (1+r)\frac{1+\tau_Y}{1+\tau_O} \qquad (2-6)$$

So another way of getting the previous result (that τ_O must equal τ_Y if the tax system is optimal) is to notice the following : if the tax system is optimal, the equi–proportional Ramsey rule implies that C_O and C_Y must fall by the same proportion, which means that C_O/C_Y must stay the same ; equation $(2-6)$ says that the only way that C_O/C_Y can stay the same is if $1+\tau_Y = 1+\tau_O$.

A third way to derive the optimal tax rates is to solve the problem directly : maximize the consumer's utility subject to the revenue requirement that $\tau_Y C_Y + \tau_O C_O/(1+r) = R$. From equations $(2-3)$, $(2-4)$ and $(2-5)$, the consumer's utility u equals

$$20 \ln (20 \frac{w}{1+\tau_Y}) + 20 \ln (20 \frac{w(1+r)}{1+\tau_O}) - 20 \frac{1+\tau_Y}{w} \frac{w}{1+\tau_Y} - 20 \frac{1+\tau_O}{w(1+r)} \frac{w(1+r)}{1+\tau_O}$$

which equals

$$-20 \ln (1+\tau_Y) - 20 \ln (1+\tau_O) + K \qquad (2-7)$$

where K is a constant which does not depend on τ_Y or τ_O.

($K = 40 \ln 20 + 40 \ln w + 20 \ln (1+r) - 40$)

Equations $(2-4)$ and $(2-5)$ imply that

$$R = 20 \tau_Y \frac{w}{1+\tau_Y} + 20 \tau_O \frac{w(1+r)}{1+\tau_O} \qquad (2-8)$$

So the government maximizes $(2-7)$ subject to the constraint $(2-8)$, implying it maximizes the Lagrangean function

$$-20\ln(1+\tau_Y) - 20\ln(1+\tau_O) + K + \lambda[20\tau_Y \frac{w}{1+\tau_Y} + \frac{1}{1+r} 20\tau_O \frac{w(1+r)}{1+\tau_O} - r] \qquad (2-9)$$

with respect to τ_Y and τ_O.

Taking derivatives with respect to τ_O and τ_Y and setting them equal to zero, the first–order conditions for optimality are

$$\frac{1}{1+\tau_Y} = \lambda w \frac{1}{(1+\tau_Y)^2} \qquad (2-10)$$

$$\frac{1}{1+\tau_O} = \lambda w \frac{1}{(1+\tau_O)^2} \qquad (2-11)$$

which means that $\tau_O = \tau_Y$.

*Q*3. Suppose everything is the same as in the previous question (#2) — the same preferences, the same government revenue requirement, the same wage rate and interest rate. The only difference is in the government's tax instruments. Now the only taxes it can levy are a proportional sales tax on the person's wage earnings when young, and a proportional tax on her interest income when old.

What is the relation between the wage tax rate when the person is young, and the tax rate on interest income when she is old, if the government wants to raise the required present value of revenue at minimum harm to the person?

*A*3. If wage income and interest income are taxed, let ω denote the person's wage, net of all taxes, and ρ denote the interest rate that she receives, net of all taxes.

Now her budget constraint is

$$C_O = (1+\rho)(\omega H - C_Y) \qquad (3-1)$$

so that

$$H = \frac{C_Y}{\omega} + \frac{C_O}{\omega(1+\rho)} \qquad (3-2)$$

and picking C_Y and C_O so as to maximize u, after substituting for H from $(3-2)$ implies optimal consumption levels of

$$C_Y = 20\omega \qquad (3-3)$$

$$C_O = 20\omega(1+\rho) \qquad (3-4)$$

Now comparison of equations $(3-3)$ and $(3-4)$ with equations $(2-4)$ and $(2-5)$ shows that the government, if it can tax wage and interest income, can get the consumer to behave exactly as she would if she faced tax rates τ_O and τ_Y on consumption. Setting

$$\omega = \frac{w}{1+\tau_Y} \qquad (3-5)$$

$$1 + \rho = \frac{1 + \tau_Y}{1 + \tau_O}(1 + r) \tag{3-6}$$

makes $(3-3)$ and $(3-4)$ exactly equivalent to $(2-4)$ and $(2-5)$.

How many hours would the person work if the taxes were defined by $(3-5)$ and $(3-6)$? Substituting into $(3-2)$,

$$H = \frac{1 + \tau_Y}{w} C_Y + \frac{1 + \tau_O}{w(1 + r)} C_O$$

which is exactly equation $(2-2)$. So defining ω and ρ by $(3-5)$ and $(3-6)$ would give exactly the same (C_Y, C_O, H) combination as using consumption tax rates τ_O and τ_Y. The present value of the tax revenue collected by any tax system is

$$wH - C_Y - \frac{C_O}{1 + r} \tag{3-7}$$

since expression $(3-7)$ is the value of the output produced by the person, minus the present value of her consumption expenditures.

So the consumption tax systems (τ_Y, τ_O), and the income tax system defined by $(3-4)$ and $(3-5)$ not only give rise to the same consumer choices, they yield the same present value of tax revenues. That means that the problem of choosing the optimal income tax rates on wage and interest income is identical to the problem of choosing the optimal consumption tax rates.

In question 2, the optimal tax system was one in which $\tau_O = \tau_Y$. From equation $(3-6)$, this means that the optimal income tax system should have $\rho = r$, meaning that there should be no tax on interest income. Here the optimal income tax is a tax on wage income alone, in which the consumer's after–tax interest rate equals the market interest rate r.

This result can be obtained directly as well. The consumer's utility, when her net wage is ω, and when her net return on saving is ρ, is (from equations $(3-1)$, $(3-2)$ and $(3-3)$)

$$20 \ln (20\omega) + 20 \ln (20\omega(1 + \rho)) - 40 \tag{3-8}$$

The present value of the tax revenue raised is

$$R = 40(w - \omega) + 20\frac{\rho - r}{1 + r}\omega \tag{3-9}$$

since the government collects $w - \omega$ dollars in wage income tax per hour worked and $r - \rho$ dollars per dollar saved, and equation $(3-4)$ implies that the person saves 20ω.

Maximizing $(3-8)$ subject to $(3-9)$ implies maximizing the Lagrangean

$$40 \ln 20 + 40 \ln \omega + 20 \ln (1 + \rho) + \lambda[40(w - \omega) + 20\frac{\rho - r}{1 + r}\omega - R] \tag{3-10}$$

with respect to ω and ρ.

First–order conditions for optimality are

$$\frac{40}{\omega} = 20\lambda[2 - \frac{\rho - r}{1 + r}] \tag{3-11}$$

$$\frac{20}{1+\rho} = 20\lambda \frac{1}{1+r}\omega \tag{3-12}$$

Equation $(3-12)$ implies that

$$\lambda = \frac{1+r}{1+\rho}\frac{1}{\omega} \tag{3-13}$$

Substitution from $(3-13)$ for λ in $(3-11)$ yields

$$3\frac{\rho - r}{1+r} = 0$$

or $\rho = r$. Since the tax on the return to saving is $r-\rho$ per dollar saved, $\rho = r$ means that the return to saving should not be taxed in this example, and that financing the entire revenue requirement from a tax on wage income is optimal.

*Q*4. Suppose that the populace of a small imaginary country consists of two groups of people. "High–ability" people, comprising 40% of the country's work force, can earn an annual income of 60, whereas the other 60% can earn an annual income of 45.

The government must raise tax revenue averaging 12 per person, using a flat tax, in which each person's tax liabilities (whether she is high–ability or lower–ability) are

$$T \equiv \tau(Y - E)$$

where Y is her annual reported income, τ the marginal tax rate, E the exemption level, and T the person's tax liabilities. If a person reports income less than E, she pays no tax (but gets no money back from the government.)

People do not get to choose how many hours to work in this little economy. However, they can choose whether or not to work in the commercial sector, or the "cash only" sector. If they work in the "commercial" sector, they earn their regular income (60 or 45, depending on whether they are "high–ability" or not), and all income is reported to the tax authorities. If they work in the "cash only" sector, they make only half as much money (30 or 22.5, depending on whether or not they are "high–ability"). But none of their income from the "cash only" sector gets reported to the tax authorities.

So each person has to choose one sector or the other, and chooses whichever job gives her the highest net income.

In this economy, which choice of flat tax system (τ, E) would be best for the ("low–ability") majority?

*A*4. Here the high–ability people will choose to work in the cash–only sector if and only if it yields them a higher net income than working in the commercial sector. Working in the commercial sector earns them 60, minus their taxes. Working in the cash–only sector earns them 30. So they will be willing to work in the commercial sector only if their total taxes (if they work in the commercial sector) are 30 or less.

So 30 is the most tax revenue (per person) which can be collected from high–ability people. Any attempt to collect more tax revenue from them would just drive them out of the commercial sector, and result in no taxes at all being collected from them.

If the tax system collects 30 from each high–ability person, how much must it collect from each other person? The required tax yield is 12 per person. So if the tax system collects T_0 dollars from each low–ability person, then the average tax revenue collected per person would be

$$(0.4)(30) + (0.6)(T_0) \qquad (3-1)$$

since 40 percent of the population is of high ability. If $(0.4)(30) + (0.6)(T_0) = 12$, then

$$T_0 = \frac{(5)(12)}{3} - \frac{2}{3}(30)$$

or

$$T_0 = 0$$

So it appears that the best that the low–ability people can do is to have a tax system which collects all the required tax revenue from the high–ability people, and collects no tax from the lower–ability people. But might they want a system which collected less than 30 from each high–ability person? No, because then the tax system would have to collect some money from them.

So the best tax system for the low–ability majority here is one in which each of them pays no taxes, and in which each of the high–ability people pays 30.

What tax system is that? The marginal rate τ and the exemption level E must satisfy

$$\tau(60 - E) = 30 \qquad (3-2)$$

$$\tau(45 - E) = 0 \qquad (3-3)$$

Equation $(3-3)$ says that E must equal 45. Then $(3-2)$ implies that $\tau = 2$. In this case, the system has an exemption of 45, and a marginal tax rate of 200 percent on all income above 45!

A marginal tax rate greater than 200 percent is usually a pretty bad idea. But here I had assumed that people did not choose how many hours they worked, which means that they could not respond to the outrageous marginal rate by working less.

The above answer assumed that working in the commercial or cash sector was an all–or–nothing decision. If instead people could work some time in each sector, then the marginal rate τ could not exceed 50 percent : when the marginal rate is greater than 50 percent, each additional hour in the commercial sector pays less (net of tax) than working for cash. (If $\tau > 0.5$, then high–ability people would work a fraction $E/60$ of their time in the commercial sector, and $1 - E/60$ in the cash sector, reporting income of E, paying no taxes, and having an after–tax income of $E + (60 - E)/2 = 30 + E/2$.)

So, if people could spend some time in each sector, then the marginal tax rate could not exceed 50 percent. The tax revenue collected per person would be

$$(0.5)[(0.4)(60 - E) - (0.6)(45 - E)]$$

which equals

$$(0.5)(51 - E)$$

Given the revenue requirement of 12 per person, $(0.5)(51 - E) = 12$, or

$$E = 27$$

If people could spend part of the working day in each sector, the best the low–ability people could do would be a system with $\tau = 0.5$ and $E = 27$, so that each low–ability person paid taxes of 9 : any higher τ would result in the high–ability people shifting enough work to the cash sector so as to avoid paying any taxes, and any lower τ would result in higher taxes for low–ability people.

*Q*5. According to the Haig–Simons (or "comprehensive") definition of income, what would the annual taxable income be for the following person?

She earned $90,000 in salary. Of that salary, $5,000 went into a company pension plan. In addition, her employer contributed $5,000 into her account in the company pension plan.

She owns her own house, which was worth $500,000 at the beginning of the year, and $600,000 at the end of the year. Her annual property taxes on the house were $10000. She spent $10000 a year on maintenance, utilities and insurance on the house. She also has a $300,000 mortgage on the house, on which she paid $15,000 in interest. She estimates that the house would rent for $45,000 a year if it were rented to someone else.

She leases a car for $10,000 a year, and spends another $4000 on insurance, gasoline, and maintenance for the car. She drives 20,000 kilometres per year, 10,000 on trips to and from work, and 10,000 on trips for shopping or entertainment.

*A*5. The Haig–Simons definition of income is the amount that a person could spend on consumption in the year, without changing the value of her wealth. So an employer's contribution to a pension plan is part of Haig–Simons income : her pension plan benefits are part of his wealth ; company contributions increase her wealth ; therefore she could increase her consumption expenditure without changing her wealth.

Her own contributions, from her own income, to a pension plan, do not affect her Haig–Simons income. Haig-Simons income is unaffected by how the income is allocated between consumption and saving.

Under the Haig–Simons definition of income, the "imputed" rent from living in her house must be included as part of her income : it is part of the value of her consumption. The value to her of getting to live in her own house is the amount of annual rent that someone would pay to live in this house. But, under Haig–Simons principles, any expenses (taxes, maintenance, or mortgage interest) of owning a house would be deductible from the "imputed" income from living in the house.

Since the value of her house is part of her wealth, any increase in that value is part of her Haig–Simons income, to be included in the year in which the increase occurs.

Under Haig–Simons principles (but not under current CRA rules), any expenses incurred to earn taxable income should be deductible from taxable income. That means in this case that half of her annual car ownership costs should be deductible.

Therefore her Haig–Simons income would be the \$90,000 in salary, plus the employer's contribution of \$5000, plus the capital gain of \$100,000 on her house, plus the imputed rental income of \$45,000 from living in the house, minus the expenses \$10,000 for taxes, \$10,000 for maintenance, and \$15,000 for mortgage interest, minus work–associated automotive expenses of $(10000+4000)/2$, for a total of \$198,000.