*Q*1. If a person's preferences can be represented by a utility function

$$u(Y, Z) = Z + 4\sqrt{Y}$$

where Y and Z are the person's quantities consumed of clothing and other goods, and if the net–of–tax price of each of the goods is 1, and if the person's income is 8, what would be the total excess burden of a unit tax of \$1 on clothing (good Y)?

*A*2. To answer this question, first it is necessary to find the demand function for the taxed good, good Y. To do so, consider the consumer's problem, to find the highest possible indifference curve in her budget set. Solving this problem means find the consumption bundle which is on her budget line, and where the slope of her indifference curve (her marginal rate of substitution) equals the price ratio.

This condition for optimality can be written

$$MRS \equiv \frac{u_Y}{u_Z} = \frac{P_Y}{P_Z} \qquad (1-1)$$

where u_Y and u_Z are her marginal utilities of clothing and other goods respectively, and where P_Y and P_Z are the prices she must pay for the two goods (including any taxes).

Here

$$u_Y = \frac{2}{\sqrt{Y}}$$

$$u_Z = 1$$

so that equation $(1-1)$ becomes

$$\frac{2}{\sqrt{Y}} = \frac{P_Y}{P_Z} \qquad (1-2)$$

or

$$Y = 4[\frac{P_Z}{P_Y}]^2 \qquad (1-3)$$

Equation $(1-3)$ is her demand function for clothing. This equation represents both the uncompensated ("ordinary") demand curve, and the compensated demand curve. Here, preferences are *quasi–linear*, so that her demand for clothing does not depend on her income ; there is no income effect, and the compensated and uncompensated demand curves are the same.

Initially, the prices of clothing and of other goods are both 1 ; the tax increases P_Y from 1 to 2. So

$$Y_0 = 4[\frac{1}{1}]^2 = 4 \qquad (1-4)$$

$$Y_1 = 4[\frac{1}{2}]^2 = 1 \qquad (1-5)$$

are her demands for clothing when there is no tax (Y_0), and when there is a \$1 unit tax on clothing (Y_1).

The tax revenue is $t_Y Y_1$, the unit tax times the quantity of clothing she buys (when clothing is subject to tax). From equation $(1-5)$,

$$TR = 1$$

when $t_Y = 1$.

The excess burden is the compensating variation to the tax increase, minus the tax revenue. There are (at least) two ways to calculate the compensating variation CV. One is to use the "area inside the compensated demand curve" formula :

$$CV = \int_{P_0}^{P_1} Y^c(P)dP$$

where P_0 and P_1 are the initial (no tax) and final (tax included) price of clothing, and Y^c is the equation of her compensated demand curve. Here $P_0 = 1$, $P_1 = 2$ and $Y^c(P) = 4/P^2$.

So

$$CV = \int_1^2 \frac{4}{P^2} dP \tag{1-6}$$

since the integral is the reverse of the derivative, $-4/P$ is the integral of the function $4/P^2$, so that

$$CV = -[\frac{4}{2} - \frac{4}{1}] = 2 \tag{1-7}$$

Another way to compute the compensating variation is to ask directly how much income the person would need to compensate her for the price increase. To do this, her consumption of other goods must be calculated. Her demand for clothing is $4/(P_Y)^2$ when the price of clothing is P_Y, so what she spends on clothing is $P_Y Y = 4/P_Y$. Since she has \$8 to spend in total, that leaves her with

$$8 - 4/P$$

to spend on other goods. So her (uncompensated) demand function for other goods is

$$Z = 8 - 4/P_Y \tag{1-8}$$

which means that her consumption of other goods is

$$Z_0 = 8 - \frac{4}{1} = 4$$

$$Z_1 = 8 - \frac{4}{2} = 6$$

when there is no tax, and when clothing is taxed, respectively.

So her utility initially is

$$U^0 = Z_0 + 4\sqrt{Y_0} = 12 \tag{1-8}$$

$$U^1 = Z_1 + 4\sqrt{Y_1} = 10 \tag{1-9}$$

How much compensation would she need, after the tax has been imposed on clothing, to get her utility back up to U^0? Since she spends all her increases in income on other goods, and since the price of other goods is 1, if she gets \$2 more in income, she will increase Z from 6 to 8, and her utility will go back up to 12. Therefore, it will take \$2 in compensation to get her back to her pre–tax level of utility.

Both methods give the same answer : the compensating variation is \$2, so that the excess burden from the tax is

$$EB = CV - TR = 2 - 1 = 1$$

*Q*2. Suppose that a person's preferences can be represented by a utility function

$$u(Y, Z) = Z - \frac{1}{X} + 2\sqrt{Y}$$

where X, Y and Z are the person's quantities consumed of food, clothing and other goods, if the net–of–tax price of each of the goods is 1, and if the person's income is 8

If the government could not tax good Z, but could choose whatever tax rates it wanted on goods X and Y, what would be the relative tax rates on the two goods?

*A*1. As in question #1, the first step here is to find the demand functions for the taxed goods. A consumer, facing prices P_X, P_Y and P_Z for the 3 goods, will choose quantities X, Y and Z to reach the highest possible indifference curve in her budget set, so that she will want a consumption bundle (X, Y, Z) for which the MRS between any two goods equals their price ratio, or

$$MRS_{XZ} \equiv \frac{u_X}{u_Z} = \frac{P_X}{P_Z} \tag{2-1}$$

$$MRS_{YZ} \equiv \frac{u_Y}{u_Z} = \frac{P_Y}{P_Z} \tag{2-2}$$

Here

$$u_X = \frac{1}{X^2} \quad ; \quad u_Y = \frac{1}{\sqrt{Y}} \quad ; \quad u_Z = 1$$

so that the optimality conditions $(2-1)$ and $(2-2)$ become

$$\frac{1}{X^2} = \frac{P_X}{P_Z} \tag{2-3}$$

$$\frac{1}{\sqrt{Y}} = \frac{P_Y}{P_Z} \tag{2-4}$$

or

$$X = \sqrt{\frac{P_Z}{P_X}} \tag{2-5}$$

$$Y = [\frac{P_Z}{P_Y}]^2 \tag{2-6}$$

Equations $(2-5)$ and $(2-6)$ are the demand functions for goods X and Y. As in question #1, the preferences here are *quasi–linear*, so that the quantities demanded of goods X and Y do not vary with the person's income ; equations $(2-5)$ and $(2-6)$ represent both the uncompensated ("ordinary") demand functions for the 2 goods, and the compensated demand functions.

Another special feature of these demand functions : quantity demanded of good X does not depend on the price of good Y, and quantity demanded of good Y does not depend on the price of good X. The quantities demanded of each good are independent of the price of the other taxed good.

This independence means that the "inverse elasticity" version of the Ramsey rule can be used to derive the optimal commodity taxes, if only goods X and Y are to be taxed.

Here

$$\epsilon^c_X \equiv -\frac{\partial X}{\partial P_X}\frac{P_X}{X} = \frac{1}{2}(P_X)^{-1.5}(P_Z)^{0.5}P_X\sqrt{\frac{P_X}{P_Z}} = \frac{1}{2} \tag{2-7}$$

$$\epsilon^c_Y \equiv -\frac{\partial Y}{\partial P_Y}\frac{P_Y}{Y} = 2(P_Z)^2(P_Y)^{-3}P_Y\frac{(P_Y)^2}{(P_Z)^2} = 2 \tag{2-8}$$

The inverse elasticity rule says that the tax rates (as a percentage of the prices) of goods X and Y should be inversely proportional to the compensated own–price elasticities of demand for the 2 goods. Since $\epsilon^c_Y = 4\epsilon^C_X$, therefore, the tax rate on food (good X) should be 4 times as large as the tax rate on clothing (good Y).

[You can check, using the demand functions defined in $(2-5)$ and $(2-6)$, that if the price of good X goes up 4 times as much as the price of good Y, then the quantities demanded of the 2 goods will fall by the same proportion, so that the "equi–proportional" version of the Ramsey rule gives the same answer.]

*Q*3. If good Z cannot be taxed, and if the world ("before tax") prices of goods X, Y and Z are

$$p_X = 2 \quad ; \quad p_Y = 2 \quad ; \quad p_Z = 1$$

and if the compensated demand functions for goods X and Y are

$$X = 225[\frac{P_Z^2}{P_Y P_X - P_Z^2}]^2$$

$$Y = 225[\frac{P_Z P_X}{P_Y P_X - P_Z^2}]^2$$

then what would be the tax rate on good X, if good Y were taxed at a rate of 50 percent (as a fraction of the before–tax price p_Y), if the tax system were optimal? Explain briefly.

$A3$.Here, the quantity demanded of good X depends on the price of good Y, and the quantity demanded of good Y depends on the price of good X, so that the "inverse elasticity" version of the Ramsey rule will **not** be useful.

So the "equi–proportional" version of the Ramsey rule must be used. If the tax system is optimal, the quantities demanded of each good must fall by the same proportion.

If both X and Y fall by the same proportion, then it must be true that the fraction $\frac{X}{Y}$ will not be changed by the taxes.

From the definitions of the compensated demand functions,

$$\frac{X}{Y} = \frac{P_Z}{P_X} \tag{3 – 1}$$

Good Z is not taxed, so that P_Z remains at 1, before and after the tax. So, if the taxes on goods X and Y are optimal, it must be the case that X/Y does not change, so that P_Z/P_X doesn't change. If $P_Z = 1$, that means that P_X cannot change.

So the optimal commodity tax system — if good Z cannot be taxed — is not to tax good Z at all.

Checking, when there are no taxes, so that $P_X = 2 = P_Y$ and $P_Z = 1$, then

$$X_0 = \frac{225}{(4-1)^2} = 25 \tag{3 – 3}$$

$$Y_0 = \frac{900}{(4-1)^2} = 100 \tag{3 – 4}$$

If good Y is taxed at 50 percent, so that $P_Y = (1.5)2 = 3$, and if good X is not taxed at all, then

$$X_1 = \frac{225}{(6-1)^2} = 9 \tag{3 – 5}$$

$$Y_1 = \frac{900}{(6-1)^2} = 36 \tag{3 – 6}$$

so that quantities demanded of both goods are reduced by the tax, by the same percentage, 64 percent.

$Q4$. An economy consists of 3 million people. Each person has the same preferences over her consumption C and the number of hours she works per week H, represented by the utility function

$$U(C, H) = C - H^2$$

Each person's wage depends on her productivity (which is exogenous, and not affected by government policy). One million people each earn a wage (before any tax deductions) of \$10 per hour ; one million people each earn a wage of \$20 per hour ; the remaining one million people each earn a wage of \$50 per hour. Each person chooses how many hours she wishes to work. Her net wage income is spent on consumption C.

If the government taxes all labour income at a rate τ, how does the government tax revenue per person vary with the tax rate τ?

*A*4. The first step is to determine how many hours a person will choose to work.

If she earns a wage of w per hour, and if that wage income is taxed at the rate τ, then her net wage (after taxes have been deducted) is $w(1-\tau)$ per hour. Her consumption is

$$C = w(1-\tau)H$$

if she chooses to work H hours.

So she should choose hours worked so as to maximize her utility,

$$C - H^2 = w(1-\tau)H - H^2 \tag{4-1}$$

Maximizing expression $(4-1)$ with respect to H,

$$w(1-\tau) - 2H = 0 \tag{4-2}$$

so that her labour supply is

$$H = \frac{w(1-\tau)}{2} \tag{4-3}$$

The total (before any taxes are deducted) income of this person is her hours worked times her wage per hour, or

$$I = w[\frac{w(1-\tau)}{2}] = (1-\tau)\frac{w^2}{2} \tag{4-4}$$

The total taxes collected from the person are τ times her income, or

$$\tau(1-\tau)\frac{w^2}{2}$$

Since the different types of people's wages are \$10, \$20 and \$50, then w^2 is 100 for the 1 milllion low–wage people, 400 for the 1 million medium–wage people, and 2500 for the 1 million high–wage people.

So the total tax revenue collected, if the tax rate is τ, will be

$$TR = 1000000\tau(1-\tau)\frac{1}{2}[100 + 400 + 2500] = 1500000000\tau(1-\tau) \tag{4-5}$$

Notice that tax revenue at first will rise with the tax rate τ, as τ increases above 0, but eventually it will fall. The function $f(\tau) = \tau(1-\tau)$ increases with τ until $\tau = 0.5$ (at which point $f'(\tau) = 0$), but then falls ; at high tax rates, further increases in τ reduces people's labour supply so much that tax revenue actually falls.

*Q*5. In the country described in question #4, if the government tax revenue from the labour income tax were distributed (in cash) equally to all 3 million people, which tax rate would people earning a wage of \$20 per hour prefer most?

*A*5. Suppose that the tax rate were τ. From equation $(4-5)$, the tax revenue per person will be

$$tr = 500\tau(1-\tau) \tag{5 − 1}$$

since there are 3 million people in total.

Also, from equation $(4-3)$, a person of wage w would choose to work $(1-\tau)\frac{w}{2}$ hours if the tax rate were τ, and would therefore have a net–of–tax wage income of

$$\frac{1}{2}w^2(1-\tau)^2$$

since she earns $w(1-\tau)$ per hour, net of taxes. Since she also receives her share tr of the tax revenue, her total income available for consumption will be

$$C = \frac{1}{2}w^2(1-\tau)^2 + 500\tau(1-\tau) \tag{5 − 2}$$

Her overall utility is $C - H^2$, or

$$U = \frac{1}{2}w^2(1-\tau)^2 + 500\tau(1-\tau) - \frac{1}{4}w^2(1-\tau)^2 = \frac{1}{4}w^2(1-\tau)^2 + 500\tau(1-\tau) \tag{5 − 3}$$

The tax rate which she will prefer is the rate which maximizes expression $(5-3)$. Differentiating $(5-3)$ with respect to τ, and setting the derivative equal to 0,

$$500(1-2\tau) - \frac{1}{2}w^2(1-\tau) = 0 \tag{5 − 4}$$

For a person earning \$20 per hour, equation $(5-4)$ becomes

$$500 - 1000\tau - 200 + 200\tau = 0 \tag{5 − 5}$$

or

$$\tau = \frac{3}{8} = 0.375 \tag{5 − 6}$$

Even though the tax which maximizes the tax revenue is 50 percent, this person would prefer a lower tax rate, since she cares about her net wage per hour, as well as the tax revenue she receives.