AS/ECON 4070 3.0AF     Answers to Assignment 2     November/February 2008

Do **all 5** questions. All count equally

*Q*1. Suppose that a person's compensated demand functions for three goods were

$$X = \frac{60}{P_X}$$

$$Y = 120\frac{P_Z}{P_Y - P_Z}$$

$$Z = \frac{120}{P_Z(P_Y - P_Z)}$$

where $X$, $Y$ and $Z$ are the quantities demanded of the three goods, and $P_X$, $P_Y$ and $P_Z$ are the prices — including any taxes — paid by the consumer for the goods.

(This consumer does consume more goods than just $X$, $Y$ and $Z$ : but these other goods cannot be taxed.)

If the world prices of goods $X$, $Y$ and $Z$ were 4, 12 and 4 respectively, then what would the optimal tax rates be if the government needed to raise revenue of 160, and could tax only goods $Y$ and $Z$?

*A*1. This is a problem in optimal commodity taxation, since the problem is to choose tax rates on commodities $Y$ and $Z$.

The simplest rule for optimal commodity taxation is the "inverse elasticity" rule. However, this is **not** a good rule to apply in this particular problem. Why not? The inverse elasticity rule applies only when the quantities demanded of one taxed good does not depend on the price of the other taxed good. Here the quantity demanded of good $Y$ depends on the price $P_Z$ (and the quantity demanded of $Z$ depends on $P_Y$).

So the best rule to use is the "equi–proportional" rule : the optimal tax system should reduce demand for goods $Y$ and $Z$ by the same proportion.

If the quantities $Y$ and $Z$ were to fall by the same proportion, then the ratio $Y/Z$ would be unchanged. (For example, if $Y$ and $Z$ both fell by 20 percent, then the ratio of quantities consumed of the two goods would go from $Y/Z$ to $(0.8)Y/(0.8)Z$, which means this ratio has not changed.)

So the equi–proportional Ramsey rule says that the optimal commodity tax system should leave the ratio $Y/Z$ the same as it was before there were any taxes at all.

Now from the equations given for the compensated demand functions,

$$\frac{Y}{Z} = (P_Z)^2 \qquad (1-1)$$

Equation $(1-1)$ says that the ratio of consumption of these two goods happens to depend only on the price of good $Z$. In particular, the ratio $Y/Z$ will stay the same if and only if $P_Z$ stays the same.

So the optimal tax system should tax **only good** $Y$.

How much should the tax be on good $Y$? Let $t_Y$ be the unit tax on good $Y$. If initially $p_Y = 4$ and $p_Z = 12$, then the price $P_Y$ paid by consumers for good $Y$ will be $4 + t_Y$. Therefore, the tax revenue collected from the tax — if good $Z$ is not taxed — is

$$TR = t_Y Y = 120 t_Y \frac{4}{12 + t_Y - 4} \tag{1-2}$$

Since the tax must raise total revenue of 48, the unit tax $t_Y$ must satisfy the equation

$$120 t_Y \frac{4}{12 + t_Y - 4} = 160 \tag{1-3}$$

or

$$160(8 + t_Y) = 480 t_y \tag{1-4}$$

which means that

$$t_Y = 4$$

Expressed as an *ad valorem* tax, the tax rate on good $Y$ should be $4/12 = 1/3$, expressed as a fraction of the before–tax price.

Checking that the tax rates (expressed as fractions of the before–tax prices) $\tau_Y = \frac{1}{3}$ and $\tau_Z = 0$ are optimal : when there were no taxes, so that $P_Y = 12$, $P_Z = 4$, then

$$Y = 120 \frac{4}{12 - 4} = 60 \tag{1-5}$$

$$Z = 120 \frac{1}{4(12 - 4)} = 3.75 \tag{1-6}$$

With a unit tax of 4 on good $Y$, and no tax on good $Z$

$$Y = 120 \frac{4}{16 - 4} = 40 \tag{1-7}$$

$$Z = 120 \frac{1}{4(16 - 4)} = 2.5 \tag{1-8}$$

so that

$$\frac{\Delta Z}{Z} = \frac{1.25}{3.75} = \frac{1}{3} = \frac{20}{60} = \frac{\Delta Y}{Y}$$

so that the "equi–proportional" Ramsey rule is satisfied.

*Q*2. Suppose that the consumer's demands were the same as they were in question #1 above, but that the government could tax only goods $X$ and $Y$ (but not good $Z$).

If the tax rates on $X$ and $Y$ were set optimally, and if the tax on good $Y$ (expressed per unit) were \$4 per unit (that is, 25%, expressed as a fraction of the tax–included price), then what must the tax rate be on good $X$?

*A*2. As shown in the answer to question #1, a tax of \$4 per unit on good $Y$ reduces demand for the good by 1/3, from 60 to 40.

If the tax rates on goods $X$ and $Y$ are optimal, then the tax on good $X$ must also lower demand for $X$ by 1/3. Initially, demand for $X$ is

$$X = \frac{60}{4} = 15 \tag{2-1}$$

So, if the tax system is optimal, then the demand for good $X$ must also be reduced by 1/3, to 10. So the unit tax $t_X$ on good $X$ should satisfy the condition

$$\frac{60}{4 + t_X} = 10$$

or

$$t_X = 2$$

Alternatively, demands for goods $X$ and $Y$ are independent of each others' prices. ($\partial X/\partial P_Y = \partial Y/\partial P_X = 0$.) So the inverse elasticity rule can be used here. The own–price elasticity $\eta^X$ of demand for good $X$ is 1. Evaluated at the after–tax price $P_Y = 16$, the own–price elasticity of demand for good $Y$ is

$$\eta_Y = -\frac{\partial Y}{\partial P_Y}\frac{P_Y}{Y} = \frac{480}{(P_Y - 4)^2}\frac{16}{40} = \frac{480}{(12)^2}\frac{16}{40} = \frac{4}{3} \tag{2-2}$$

So the tax rate on good $X$ should be 4/3 of the tax rate on good $Y$, since $\eta_Y/\eta_X = \frac{3}{4}$. Expressed as a fraction of the before–tax price, $\tau_Y = 4/12 = 1/3$, so that the inverse elasticity rule implies that

$$\tau_X = \frac{4}{3}\frac{1}{3} = \frac{4}{9} \tag{2-3}$$

That's not exactly the answer $\tau_X = \frac{1}{2}$ obtained using the equi–proportional rule, because the formulae are approximations, and because the own–price elasticity of demand for good $Y$ is not constant. (Evaluated at the prices when there were no taxes, $\eta_Y = \frac{480}{(8)^2}\frac{12}{60} = \frac{3}{2}$, which would imply an optimal $\tau_X$ of $\frac{1}{2}$.)

*Q*3. Suppose now that the consumer's demands were still the same as in question #1 above, but now the government can tax all 3 goods $X$, $Y$ and $Z$.

If the tax rates on $X$ and $Y$ were set optimally, and if the tax on good $Y$ (expressed per unit) were \$4 per unit (that is, 25%, expressed as a fraction of the tax–included price), then what must the tax rates be on good $X$ and $Z$?

*A*3. Here just combining the answers to questions #1 and #2 is all that's needed. The main result in the answer to question 1 was that good $Z$ should not be taxed, if the equiproportionality rule $\Delta Z/Z = \Delta Y/Y$ was to be satisfied. This result — $\tau_Z = 0$ — still holds, no matter how many other goods are taxed, since keeping $Y/Z$ constant requires $P_Z$ to stay constant, no matter what happens to other prices.

So optimality requires that $\tau_Z = 0$. And if $t_Y = 4$ and $t_Z = 0$, question #3 reduces to question #2 : the optimal tax on good $X$ should be 50% (expressed as a fraction of the net–of–tax price).

So $\tau_X = 0.5, \tau_Z = 0$ are the optimal tax rates (as proportions of the net–of–tax prices) if $\tau_Y = \frac{1}{3}$ and the taxes on $X$, $Y$ and $Z$ have been set optimally.

*Q*4. In an imaginary country, people all have the same preferences, which can be represented by the utility function

$$u(x, h) = x - \frac{1}{2}h^2$$

where $x$ is the dollar amount of the person's weekly consumption expenditure, and $h$ is the number of hours she works per week.

People differ only in the wage $w$, which they can earn (before any taxes are deducted) per hour. People can choose how many hours $h$ they work, and make this choice so as to maximize their utility, given that the value $x$ of their consumption depends on how much they work. The value $x$ of their consumption equals their wage income, net of any taxes, plus any money they get from the government.

In this economy, (*i*) the mean (average) value of the hourly wage $\bar{w}$ is 30, (*ii*) the variance of the hourly wage is 300, and (*iii*) the mean (average) value of $w^2$ is 1200. (This third number follows from the fact that, for any random value, the expected value of $w^2$ equals $[\bar{w}]^2$ plus the variance of $w$.)

The government is considering a negative income tax scheme, in which each person will pay taxes of $\tau y - T$, if her gross weekly earnings are $y$. (So a person whose earnings are so low that $\tau y < T$ would actually get a cheque of $T - \tau y$ from the government.

If the government has no outside revenue requirements, so that the only constraint on its negative income tax is that the tax collect a total revenue from all people which is greater than or equal to zero, how does $T$ vary with the tax rate $\tau$?

*A*4. In this economy, people's hours worked will depend on the tax rates. A person's net weekly income will be

$$x = w(1 - \tau)h + T \qquad (4-1)$$

if she were to choose to work $h$ hours per week — this is just her gross income $wh$ minus her taxes $twh - T$. So she picks $h$ to maximize

$$x - \frac{1}{2}h^2 = w(1 - \tau)h + T - \frac{1}{2}h^2 \qquad (4-2)$$

Maximizing $(4-2)$ with respect to $h$ implies

$$w(1 - \tau) = h \qquad (4-3)$$

which defines the person's labour supply curve : for these people, increases in the hourly net–of–tax wage lead to increases in the amount they choose to work.

A person's gross income equals her gross hourly wage, times the number of hours she works. If $y = wh$ denotes her gross income, then $(4-3)$ implies that

$$y = w^2(1-\tau) \qquad (4-4)$$

So the total tax revenue that the government collects from a person, if her wage is $w$, and if she faces a marginal tax rate of $\tau$, and gets a grant of $T$ is

$$TR(w) = \tau y - T = \tau(1-\tau)w^2 - T \qquad (4-5)$$

What is the average tax revenue collected per person in this country? Since the tax revenue depends on $w^2$, then the average tax revenue collected per person is the average of $TR(w)$ over the whole population, which is

$$ATR = \tau(1-\tau)(1200) - T \qquad (4-6)$$

since 1200 is the average value of $w^2$.

The only constraint on the government's tax policy is that the average tax revenue be non–negative : $ATR \geq 0$. Therefore, the maximum grant $T$ which it can pay to people, if the tax rate is $\tau$ is defined (from equation $(4-6)$) by

$$T = \tau(1-\tau)(1200) \qquad (4-7)$$

Equation $(4-7)$ defines a relation between the marginal tax rate and the grant $T$ : for low values of $\tau$, $T$ increases with the marginal tax rate. But $T$ reaches a maximum at $\tau = 0.5$ : higher tax rates than 50 percent could never be justified in this economy since they lower everyone's net wage income, and lower the grant as well.

*Q*5. In the economy described in question #4 above, what tax rate $\tau$ would be best for a person whose hourly wage $w$ was 30?

*A*5. Suppose that the government levies a tax rate of $\tau$. From the answer to question #4, the total grant that the government would pay would then be

$$T = \tau(1-\tau)(1200) \qquad (5-1)$$

A worker with wage $w$ would face a net wage of $w(1-\tau)$, and would choose to work $h = w(1-\tau)$ hours per week (from equation $(4-3)$). So her net income after taxes and transfers, available for spending on consumption, would be

$$x = w(1-\tau)h + T = [w(1-\tau)]^2 + T = [w(1-\tau)]^2 + \tau(1-\tau)1200 \qquad (5-2)$$

Her utility is $x - \frac{1}{2}h^2$ ; from $(5-2)$ and $(4-3)$, then,

$$u = x - \frac{1}{2}h^2 = [w(1-\tau)]^2 + \tau(1-\tau)1200 - \frac{1}{2}[w(1-\tau)]^2 = \frac{1}{2}[w(1-\tau)]^2 + \tau(1-\tau)1200 \quad (5-3)$$

Equation $(5-3)$ shows that, as long as $\tau$ is less than 50 percent, increasing the tax rate has two opposite affects on the person's well–being : her net wage $w(1-\tau)$ falls, which makes her worse off, but the grant $T$ which she receives increases, which makes her better off.

How does her well–being change when the tax rate changes? Differentiation of $(5-3)$ yields

$$\frac{\partial u}{\partial \tau} = -w^2(1-\tau) + (1-2\tau)1200 \quad (5-4)$$

As long as $w^2 > 1200$, the person's utility increases with the tax rate $\tau$ as it increases from 0. [At $\tau = 0$, $(5-4)$ shows that $\partial u/\partial\tau = w^2 - 1200$.]

The person's most–preferred level for the tax rate $\tau$ is the one which maximizes $u$. That maximum occurs when $\partial u/\partial\tau = 0$. From $(5-4)$ the person's most preferred tax rate $\tau^*$ is the one for which

$$w^2(1-\tau^*) = (1-2\tau^*)1200 \quad (5-5)$$

or

$$(2400 - w^2)\tau^* = 1200 - w^2 \quad (5-6)$$

so that

$$\tau^* = \frac{1200 - w^2}{2400 - w^2} \quad (5-7)$$

For a person whose wage $w$ equals 30, equation $(5-7)$ implies that

$$\tau^* = \frac{1200 - 900}{2400 - 900} = \frac{1}{5} \quad (5-8)$$

So the person's most–preferred wage tax rate is 20 percent.