Taxation and Efficiency : (a) : The Expenditure Function

The expenditure function is a mathematical tool used to analyze the cost of living of a consumer. This function indicates how much it costs — in dollars — to reach a certain standard of living.

So the value, in dollars, of the expenditure function depends on the standard of living the consumer is supposed to reach. The higher the standard of living, the more it's going to cost. In microeconomics, the standard of living is represented by an indifference curve. The higher the indifference curve, the better off is the consumer.

Let u_0 be some level of utility. That's a measure of how well–off the consumer is on some indifference curve. By definition, all consumption bundles on an indifference curve give the consumer the same level of utility. Consumption bundles on higher indifference curves give her a higher level of utility. The expenditure function, then, measures the cost, in dollars, of getting some level of utility. The higher the required utility level u_0, the higher the cost is going to be.

But the cost of living also depends on the **prices** the consumer must pay. If prices go up, then more money is required to enable the consumer to maintain the given standard of living.

Therefore, if we had two goods, food and clothing, then the expenditure function would be some function

$$E(P_F, P_C, u_0)$$

which indicates the cost, in dollars, of getting the consumer to the utility level u_0, if the price of food is P_F per kilo and the price of clothing is P_C (per shirt). More generally, if there were 100 different goods which the consumer could consume, then her expenditure function would be some function

$$E(P_1, P_2, \ldots, P_{100}, u_0)$$

indicating the cost to her of getting the utility level u_0, when the unit price of good #1 is P_1, the unit price of good #2 is P_2, and so on.

All consumption bundles on a given indifference curve will make the consumer equally well–off. So what is meant, in the previous few paragraphs, by "the cost" of getting the consumer a consumption bundle on the given indifference curve — when there are many bundles on the indifference curve? What is meant is the **lowest** cost of getting the consumer to the given indifference curve. So $E(P_F, P_C, u_0)$ is the cost of the **cheapest** bundle on the indifference curve with utility level u_0, when the price of food is P_F and the price of clothing is P_C.

What bundles are on the indifference curve? The standard tool in microeconomics is the consumer's **utility function**. A bundle containing F kilos of food and C units of clothing gives the consumer a utility level $U(F, C)$. So the bundle will give the consumer the required standard of living if the utility level $U(F, C)$ from the consumption bundle equals the required level u_0. Of course, the form of the given consumer's utility function depends on her tastes. But saying that "$U(F,C) = u_0$" is the same thing as saying that the consumption bundle (F,C) is on the indifference curve corresponding to utility level u_0.

What is the cost of a consumption bundle (F,C)? It's

$$P_F F + P_C C$$

which, of course, depends on the prices of food and of clothing. This consumer's expenditure function is derived by solving the mathematical problem of **minimizing** $P_F F + P_C C$ over the set of all consumption bundles for which $U(F,C) = u_0$.

Mathematically, this is a problem which can be solved by using the method of Lagrange multipliers : something $(P_F F + P_C C)$ is being minimized subject to a constraint $(U(F,C) = u_0)$. The method of Lagrange multipliers is to construct the expression

$$P_F F + P_C C + \lambda(u_0 - U(F,C)) \tag{1}$$

and to minimize expression (1) with respect to the choice variables F and C, and with respect to the Lagrange multiplier λ which multiplies the constraint $u_0 - U(F,C) = 0$.

Minimization of expression (1) with respect to F, C and λ means taking the derivatives of expression (1) with respect to F, C and λ, and setting all 3 derivatives equal to zero. So, minimizing the cost of the given level of utility means finding F, C and λ such that

$$P_F - \lambda \frac{\partial U}{\partial F} = 0 \tag{2F}$$

$$P_C - \lambda \frac{\partial U}{\partial C} = 0 \tag{2C}$$

$$u_0 - U(F,C) = 0 \tag{2λ}$$

The third condition (2λ) is just the constraint that the bundle of food and clothing chosen give the consumer exactly the required level of utility u_0.

If we take the first and second equations, $(2F)$ and $(2C)$, they can be written

$$\lambda = \frac{P_F}{U_F} \tag{2F$'$}$$

$$\lambda = \frac{P_C}{U_C} \tag{2C$'$}$$

where I have used the shorthand U_F and U_C for the marginal utilities of food and clothing consumption respectively.

So, from $(2F')$ and $(2C')$, we get : if F^* and C^* are the quantities of food and clothing which minimize the cost of getting to the given level of utility, then it must be true that

$$\frac{U_F}{U_C} = \frac{P_F}{P_C} \tag{3}$$

The left side of equation (3), the ratio of the marginal utilities, is the **marginal rate of substitution** between the two goods, food and clothing. It — or the negative of it — is the slope of the indifference curve.

So what does equation (3) say about how to minimize the cost of getting to a given level of utility? It says that we move along the indifference curve until the (absolute value of the) slope of the indifference curve equals the price ratio.

That is, we want to find the point (F^*, C^*) on the indifference curve where the indifference curve is tangent to a line with slope equal to the (negative of the) price ratio P_F/P_C.

Geometrically, why is (3) the solution to this cost–minimization problem? What are all the bundles (F, C) which cost exactly \$100? They are bundles for which

$$P_F F + P_C C = 100 \tag{4}$$

Equation (4) defines a line, with slope $-P_F/P_C$ (when we graph food consumption on the horizontal and clothing consumption on the vertical). And if we spend a different amount of money than \$100, say \$80, then the set of all consumption bundles which cost exactly \$80 is the set of (F, C) satisfying the equation

$$P_F F + P_C C = 80 \tag{4'}$$

which is also a line with slope $-P_F/P_C$, only further in towards the origin than the first budget line (4).

So, when we graph food consumption on the horizontal, and clothing consumption on the vertical, bundles which cost exactly Z dollars are lines with slope $-P_F/P_C$. To **minimize** the cost of the given utility level u_0, we want to find the point on the indifference curve $U(F, C) = u_0$ which is on the lowest of these cost lines $P_F F + P_C C = Z$. Geometrically, that's going to be the point on the indifference curve where the indifference curve is tangent to a line with slope $-P_F/P_C$.

Figure 1 illustrates. The thick red curve in the figure is the indifference curve corresponding to some utility level u_0. If the prices of the 2 goods are $P_F = 4, P_C = 4$, then all bundles which cost the same amount are located on lines with a slope of $-P_F/P_C = -1$. The lowest–cost bundle is the bundle where a (green) "iso–cost" line with slope -1 is tangent to the red indifference curve : at the point $(F, C) = (12, 12)$.

If the prices were different, then cost minimization (for the utility level) would lead to another point on the same indifference curve. If $P_F = 9$ and $P_C = 4$, then all bundles which cost the same are located on iso–cost lines with slope $-P_F/P_C = -9/4$. The lowest–cost bundle is the bundle where a (dark blue) iso–cost line with slope $-9/4$ is tangent to the red indifference curve : at the point $(F, C) = (8, 18)$.

Summarizing,

COST MINIMIZATION : The consumption bundle (F^*, C^*) which minimizes the cost $P_F F + P_C C$ of getting the consumer a utility level of u_0 is the consumption bundle on the indifference curve $U(F, C) = u_0$ where marginal rate of substitution U_F/U_C, which is the absolute value of the slope of the indifference curve, equals the price ratio P_F/P_C.

Now the solution to the problem just described may look very similar to that of the "standard" consumer's problem : we have an indifference curve tangent to an budget line.

But the problem presented here is sort of backwards to the standard consumer's problem. In the standard consumer problem, the consumer has a **given** amount of money M to spend, and we are looking for the consumption bundle which gets her to the highest indifference curve, given the prices and given the amount of money she has to spend. So the amount of money is given, and we find the indifference curve which is "best".

In the problem solved above, it's the **indifference curve** which is given, and it's the amount of money we're trying to solve for.

For this reason, the "standard" consumer problem, of utility maximization subject to a budget constraint, is referred to as the "primal" (or first) problem. The problem solved here, cost minimization subject to a given level of utility, is called the "dual" problem.

In solving the primal problem, we find the quantities of food and clothing which maximize the consumer's utility, given the prices, and given her income. These quantities of food and clothing are the consumer's "ordinary" demand functions. The quantities of food and clothing which she chooses (in order to maximize her utility subject to her budget constraint) depend on the prices of the goods, and on the amount of money M she has to spend. These "ordinary" demand functions are often referred to in textbooks as "uncompensated" demand functions, or "Marshallian" demand functions[1].

But the problem solved above was the **dual** problem. That's the quantities of food and clothing which **minimize** the cost of a given utility level. Again, these quantities depend on the prices of food and clothing : change P_F, for example, and the slope of the budget line $-P_F/P_C$ changes. But they also depend on the level u_0 of utility which the consumer is getting. So the F^* and C^* which solve the cost minimization problems are each dependent on P_F, P_C and on u_0. These F^* and C^* are referred to as "compensated", or "Hicksian" demand functions[2].

So I will now write the quantities F^* and C^* of food and clothing which minimize the cost of attaining the given utility levels as the "Hicksian" demand functions

$$F^H(P_F, P_C, u_0)$$

$$C^H(P_F, P_C, u_0)$$

What happened to the "expenditure function"? The expenditure function was supposed to be the cost, in dollars, of providing the consumer with a given utility level u_0. But in order to find this cost, I first had to find out what quantities of food and clothing were the cheapest way of providing the consumer with the given utility level. That's the dual problem solved above. I now know that the quantities of food and clothing which minimize the cost of getting the consumer a

[1] after the 19th–century British economist Alfred Marshall

[2] after the 20th–century British economist John Hicks

utility level of u_0 are $F^H(P_F, P_C, u_0)$ and $C^H(P_F, P_C, u_0)$ if the prices of food and of clothing are P_F and P_C.

So what's the actual cost, in dollars, of getting the consumer her utility of u_0 as cheaply as possible? It's the cost, in dollars, of the consumption bundle $(F^H(P_F, P_C, u_0), C^H(P_F, P_C, u_0)$. In other words

$$E(P_F, P_C, u_0) \equiv P_F F^H(P_F, P_C, u_0) + P_C C^H(P_F, P_C, u_0) \tag{5}$$

In Figure 1, when the prices of food and clothing were both 4, the diagram indicated that the consumer's cost of getting to the red indifference curve was minimized at the consumption bundle $(12, 12)$. In other words, in Figure 1,

$$F^H(4, 4, u_0) = 12$$

$$C^H(4, 4, u_0) = 12$$

where F^H and C^H refer to the compensated (Hicksian) demand functions for food and clothing. The cost of the consumption bundle (where the green line is tangent to the indifference curve in Figure 1) is

$$E(4, 4, u_0) = 4F^H(4, 4, u_0) + 4C^H(4, 4, u_0) = 96$$

If the price of food were to change, P_F increasing from 4 to 9 (and P_C stayed the same), then the consumption bundle which got the consumer to the same (solid red) indifference curve at lowest cost would change to the point where the new dark blue isocost line was tangent to the red indifference curve, at $(8, 18)$.

So

$$F^H(9, 4, u_0) = 8$$

$$C^H(9, 4, u_0) = 18$$

and

$$E(9, 4, u_0) = 9F^H(9, 4, u_0) + 4C^H(9, 4, u_0) = 144$$

In this example in Figure 1, the increase in the price of food — holding constant the price of clothing P_C and the consumer's utility level u_0 — had several effects.

(i) the increase in the price of food P_F **decreased** the quantity of food $F^H(P_F, P_C, u_0)$ in the solution to the cost minimization problem

(ii) the increase in the price of food P_F **increased** the quantity of clothing $F^C(P_F, P_C, u_0)$ in the solution to the cost minimization problem

(iii) the increase in the price of food P_F **increased** the cost $E(P_F, P_C, u_0)$ of getting the consumer to the indifference curve corresponding to the utility level u_0

It turns out that properties (i) and (iii) **must always** hold : increasing the price of a good must decrease the cost–minimizing quantity of that good, and must increase the cost of achieving the given utility level. Property (*ii*) must hold if there are only 2 goods, but not when there are more than two different goods[3].

In solving the problem of getting a consumer a given level of utility at the lowest possible cost, in the example in Figure 1, the quantities of goods which the consumer got turned out to **change** when a price changed. What happened is that the consumer **substituted** clothing for food when the price of food went up. This substitution is exactly the "substitution effect" which plays an important role in Econ 2300 (when the effect of a price change is divided into the substitution effect and the income effect using the Slutsky equation).

Suppose that a worker is transferred from Canada to Japan. Relative prices of goods are very different in Japan than in Canada. The price of rental housing (per square metre of space) is very high. Prices of many goods are higher in Japan. But the difference (between Japan and Canada) in the price of housing is particularly high. So the price of housing, **relative** to the price of restaurant meals, is much higher in Japan than in Canada. If this worker moves from Canada to Japan, she will figure out that she should change her consumption patterns a bit, to get the most for her money in her new location. She will substitute restaurant meals for housing : given the high relative price of residential space in Japan, she will rent a much smaller apartment than she had in Canada. Since it would be so expensive to rent a large apartment, with a big eat–in kitchen, in Japan, she will rent a smaller apartment, and use the money she saves to eat out in restaurants more.

So in order to calculate accurately the cost of living in different countries (or in the same country, in different periods), this substitution should be taken into account. How much a consumer will alter her consumption pattern in response to price changes depends on her tastes. The shape of her indifference curve shows how willing she is to change radically her consumption pattern when relative prices change. If she is unwilling to change her consumption pattern very much, she regards the different goods as being not–very–substitutable for each other. Her indifference curve will change **a lot** in slope as we move up or down. The most extreme case of unwillingness to substitute would be a person who regarded the different goods as perfect complements for each other, and who had L–shaped indifference curves. The opposite extreme, someone who found it very easy to substitute one good for another, would have indifference curves for which the slope did not change much as we moved up or down. In the extreme, perfect substitutes, the indifference curve would be a straight line.

This substitution in response to price changes makes a little more complicated the problem of calculating how much the cost of living changes when a price changes.

That is, I want to know : "what is the cost, in dollars per year, to this consumer, of having the price of food increase from \$4 per kilo to \$9 per kilo ?".

[3] That is, if there are 3 different goods which the consumer consumes, and the price of good #1 increases, with the prices of good #2 and good #3 remaining constant, then the quantity $Q_2^H(P_1, P_2, P_3, u_0)$ of good #2 in the cost–minimizing solution might increase, or it might decrease.

There is a simple answer to this question : the price of food has gone up by \$5 per kilo. If she consumes X kilos of food per year, then the cost of her food consumption has gone up from $(4)(X)$ to $(9)(X)$ when the price of food increases, so that the cost to her is the difference in her food bill : $(5)(X)$.

And there's a big problem with this simple answer : the person's food consumption does not stay **constant** when the price of food increases. In the example illustrated in Figure 1, if the person initially were on the thick red indifference curve, and if she faced a price of food of 4 (and a price of clothing of 4), then she would consume 12 units of food. When the price of food increases to \$9, she changes her food consumption from 12 to 8 — if she somehow gets to stay on the same indifference curve. So the quantity X in the previous paragraph is not a constant : it falls from 12 to 8 as the price of food increases from 4 to 9.

The expenditure function gives a precise answer to this cost–of–living problem. Actually it gives two precise answers.

Here's the first answer : let u_0 be the level of utility associated with the thick red indifference curve in Figure 1. How much money, in dollars, would the person's cost of living go up, when she is on the red indifference curve, if the price of food increased from 4 to 9? The answer to that question, which I will call "CV" is simply the change **value of the expenditure function** when the price of food goes up. That is, if the person has a utility level u_0, and if the price of clothing is \$4, then her cost of living increases by

$$CV \equiv E(9, 4, u_0) - E(4, 4, u_0) \tag{6}$$

when the price of food increases from 4 to 9.

In this example, then, I have a precise answer : her Hicksian demands were $(12, 12)$ when $P_F = P_C = 4$, and her Hicksian demands change to $(8, 18)$ when P_F increases to 9. So $CV = 144 - 96 = 48$ here.

[Notice than the "simple" answer above does provide a sort of approximation. The simple answer was that the cost of the price increase is her food consumption times the increase in the price of food. The price of food went up 5. Her food consumption was 12 "before" (when the price of food was low), and 18 "after" (when the price of food was high). So the simple answer would be $(5)(12) = 60$ if we used the "before" quantity, and $(5)(8) = 40$ if we used the "after" quantity. The correct answer, 48, lies in between.]

The term "CV" in the expression above stands for "compensating variation", because it is — exactly — the amount we would have to compensate a person for a price increase, if we want to keep her on the same indifference curve. So if a worker had the preferences represented by the indifference curves in Figure 1, and if the person were being transferred from a country where the price of food was \$4 per kilo to a country where the price was \$9 a kilo, then we would have to increase her salary by (exactly) \$48 to compensate her for the food price increase.

And equation (6) serves as an exact definition of a compensating variation. More generally, if a person has a utility level of u, and faces prices of $\mathbf{p} \equiv (p_1, p_2, \ldots, p_{100})$, then the amount, in dollars, that we would have to pay her to keep her on the same indifference curve would be

$$CV \equiv E(\mathbf{p}', u) - E(\mathbf{p}, u) \tag{7}$$

if the prices she faced changed from $\mathbf{p}$ to

$$\mathbf{p}' \equiv (p_1', p_2', \ldots, p_{100}')$$

Equation (7), which defines the compensating variation, gives a precise answer to the question : "what is the annual cost, in dollars, to this consumer of the increase in the price of food?". In equation (7) the utility level u in the expression corresponds to the indifference curve she was on before the price of food went up. That's the higher (red) indifference curve in Figure 2.

But above I mentioned that there would be not one, but **two** precise answers.

The other precise answer involves the other indifference curve shown in Figure 2 — the lower, green indifference curve.

This second precise answer is called the "equivalent variation", and is defined by

$$CV \equiv E(\mathbf{p}', u_1) - E(\mathbf{p}, u_1) \tag{8}$$

where u_1 is the utility level corresponding the the green indifference curve, the consumer's utility level if the price of food is 9, and the price of clothing is 4, and she has \$96 to spend. [The red indifference curve corresponds to her utility if the price of food is 4, and the price of clothing is 4, and she has \$96 to spend.]

That is, if $P_F = 9$ and $P_C = 4$, and the person's income were 96, then her budget line would be the dotted black line in Figure 2, with the equation $(9)F + (4)C = 96$.

What is the EV, in words? It answers the question ': "what income **reduction** would be equivalent, in terms of its effect on the consumer's utility, to the increase in the price of food?". That is, what if we gave the consumer two unpleasant choices : (i) we raise the price of food by \$5 a kilo ; (ii) we take Z dollars away from the consumer, but leave the price of food at \$4 a kilo. The equivalent variation EV is the amount of money Z which leaves the consumer exactly indifferent between the 2 options. It's the largest amount of money she would be willing to bribe an official to prevent the increase in the price of food.

In Figure 2, along the green indifference curve, the person's Hicksian demands are $F^H(4,4,u_1) = 8, C^H(4,4,u_1) = 8$ when $P_F = P_C = 4$ (and when u_1 is the level of utility of the green indifference curve), and $F^H(9,4,u_1) = 5.333, C^H(9,4,u_1) = 12$ when $P_F = 9$ and $P_C = 4$. So in Figure 2,

$$E(4,4,u_1) = (4)(8) + (4)(8) = 64 \quad ; \quad E(9,4,u_1) = (9)(5.333) + (4)(12) = 96 \tag{9}$$

which means that here

$$EV = 96 - 64 = 32$$

So my two precise answers are both precise, but they are not equal to each other : the amount we would have to compensate her for the price increase, \$48, exceeds the amount that she would be willing to bribe someone to prevent the price increase, \$32.

And both of the answers (CV and EV) are precise expressions for the cost of a tax. Suppose that some government levies a tax of \$5 on food, which previously was untaxed, and cost (previously) \$4 per kilo. Naturally, the tax on food makes food buyers worse off. The two measures correspond to the answers to the following two questions, which are slightly different from each other.

Question 1 (CV) : The government decides to introduce this tax on food. But they realize it will harm this particular consumer, and they don't want to harm the consumer (perhaps because she's a low–income consumer). So they might decide to make a cash payment to this consumer, to "repair" the damage to be done by the tax on food. How big a cash payment would they have to make, in order to exactly offset the damage done by the food tax? That is, how big does the cash payment have to be, in order for this consumer to be exactly as well off — no better, no worse — as she was before there was a food tax (and before she received the cash payment)?

The answer to this question is : "the CV to the tax".

Question 2 : A politician opposes the tax on food, and argues that taxing food is liking taking money out of this taxpayer's pocket. And we want to know exactly how much money would have to be taken from her pocket, for the damage to be as big as the damage of the food tax. That is, what loss of cash would be exactly as harmful as the food tax, if the cash loss occurred instead of the food tax?

The answer to this question is : "the EV to the tax".

Another question : how much does the cost of a tax **change** if the tax is increased slightly?

The answer is relatively straightforward. If we regard the compensating variation as the appropriate measure of the cost of a tax on food, then the cost of the tax is

$$C(t) = E(P_F^0 + t, P_C, u) - E(P_F^0, P_C, u) \tag{10}$$

where P_F^0 is the price of food when there is no tax, and t is the (unit) tax per kilo of food.

So how does the cost change when the tax changes?

From equation (10),

$$C'(t) = \frac{\partial E}{\partial P_F} \tag{11}$$

[And that would be true as well if I had used the equivalent variation : $C'(t)$ would still be the derivative of the expenditure function, just evaluated at a different utility level.]

But what is the derivative of the expenditure function with respect to the price of food? Recall the definition of the expenditure function above, and of the Hicksian demand functions :

$$E(P_F, P_C, u) = P_F F^H(P_F, P_C, u) + P_C C^H(P_F, P_C, u)$$

If this expression is differentiated with respect to P_F, then the result is

$$\frac{\partial E}{\partial P_F} = F^H(P_F, P_C, u) + [P_F \frac{\partial F^H}{\partial P_F} + P_C \frac{\partial C^H}{\partial P_F}] \tag{12}$$

But : it turns out that the expression in square brackets must equal 0. The proof of this statement is presented immediately below — but it's here only for completeness. You will not be required to know this proof.

Proof that the term in square brackets in equation (12) *is zero*

What is the change in utility, if the price of food increases? Since the person's utility can be written

$$U(F, C)$$

then the change in her utility is

$$\Delta U = \frac{\partial U}{\partial F}\frac{\partial F}{\partial P_F} + \frac{\partial U}{\partial C}\frac{\partial C}{\partial P_F} \tag{i}$$

Now, if we are looking at Hicksian, or compensated demands, these are the changes in consumption of food and clothing that keep the person on the **same indifference curve***.*

So, from equation (i)*,*

$$\frac{\partial U}{\partial F}\frac{\partial F^H(P_F, P_C, u)}{\partial P_F} + \frac{\partial U}{\partial C}\frac{\partial C^H(P_F, P_C, u)}{\partial P_F} = 0 \tag{ii}$$

Equation (3) *indicated that the Hicksian demands are defined by the tangency of the indifference curve with a budget line :*

$$\frac{U_F}{U_C} = \frac{P_F}{P_C} \tag{3}$$

which means that

$$P_F \frac{\partial F^H}{\partial P_F} + P_C \frac{\partial C^H}{\partial P_F} = \alpha[\frac{\partial U}{\partial F}\frac{\partial F^H(P_F, P_C, u)}{\partial P_F} + \frac{\partial U}{\partial C}\frac{\partial C^H(P_F, P_C, u)}{\partial P_F}] \tag{iii}$$

where the constant α *equals* $\frac{P_F}{U_F}$*.*

Equation (ii) *says that the right side of equation* (iii) *must equal zero. That means that the left side of equation* (iii) *must equal zero. But the left side of equation* (iii) *is just the expression in square brackets in equation* (12)*, which completes the proof.* •

Since the expression in square brackets in equation (12) is zero, therefore

$$\frac{\partial E(P_F, P_C, u)}{\partial P_F} = F^H(P_F, P_C, u) \tag{13}$$

Equation (13) is called *Shephard's Lemma*. It says that the change in the cost to the taxpayer when we raise a tax rate is proportional to the taxpayer's compensated demand for the taxed good.

That's not a surprising result : it says that the cost of a cigarette tax to a consumer is proportional to how many cigarettes she smokes.

But it is very convenient for the analysis of the harm done by taxes to taxpayers : it says that raising the tax on food by one cent per kilo will cost a typical consumer F cents per year, where F is the number of kilos of food she chooses to consume.

One more result is useful. Here the proof is even more esoteric, and you certainly are not required to know the proof.

RESULT : $\frac{\partial X^H(P_X,P_Y,u_0)}{\partial P_X} \leq 0$. In other words : **compensated demand curves cannot slope up**. That is, if we increase the price of a good, and compensate the person so that she stays on the same indifference curve, her quantity demanded of the good cannot increase : it either decreases or stays the same. And the latter possibility (staying the same) only happens if there are "kinks" in the indifference curves, such as would occur if goods were perfect complements.

The result is also true when there are more than two goods : the compensated demand function for a good must be a non–increasing function of the price of that good. For completeness, I include the (not–required) proof of this result below.

Proof that $\frac{\partial X^H}{\partial P_H} \leq 0$

The expenditure function comes from minimization *of the cost of a consumption bundle, subject to the utility constraint. That means that it must be a* concave *function of the prices of goods.*

If any function $f(z_1, z_2, \ldots, z_J)$ is concave, then the "Hessian" matrix of second–derivatives $\frac{\partial^2 f}{\partial z_i \partial z_j}$ must be a negative definite matrix.

In this case, then, the expenditure function being concave means that the matrix of second derivatives of the expenditure function with respect to prices, with typical element $\frac{\partial^2 E}{\partial p_i \partial p_j}$ must be negative semi–definite.

If a matrix is negative semi–definite, then the elements on the diagonal of that matrix must be negative (or 0). So the fact that the expenditure function is a concave function of prices means that

$$\frac{\partial^2 E}{\partial i \partial i} \leq 0 \tag{14}$$

for any good i.

But Shephard's Lemma says that

$$\frac{\partial E}{\partial p_i} = x_i^H(p_1, p_2, \ldots, p_n, \bar{u}) \tag{15}$$

Differentiating equation (15) *with respect to* p_i *says that*

$$\frac{\partial^2 E}{\partial p_i \partial p_i} = \frac{\partial x_i^H}{\partial p_i} \tag{16}$$

And expression (14) *therefore implies that the right side of equation* (16) *must be 0 or less.* •

One more result is useful in the next section, on optimal taxation. The "cross–partial" derivatives of Hicksian demands for one good, with respect to the price of another good, must be symmetric.

RESULT : $\frac{\partial X^H(P_X,P_Y,u)}{\partial P_Y} = \frac{\partial Y^H(P_X,P_Y,u)}{\partial P_X}$.

Proof that $\frac{\partial X^H(P_X,P_Y,u)}{\partial P_Y} = \frac{\partial Y^H(P_X,P_Y,u)}{\partial P_X}$

The Hessian matrix mentioned in the previous proof must also be symmetric. That's a mathematical result : for any function $f(x,y)$, $f_{xy} = f_{yx}$. *So* $\frac{\partial^2 E}{\partial P_X \partial P_Y} = \frac{\partial^2 E}{\partial P_Y \partial P_X}$. *But Shephard's Lemma said that* $\frac{\partial E}{\partial P_X} = X^H$ *and* $\frac{\partial E}{\partial P_Y} = Y^H$. *Therefore* $\frac{\partial Y^H(P_X,P_Y,u)}{\partial P_X} = \frac{\partial^2 E}{\partial P_X \partial P_Y} = \frac{\partial^2 E}{\partial P_Y \partial P_X} = \frac{\partial X^H(P_X,P_Y,u)}{\partial P_Y}$, *which proves the result.* •