1. A person's marginal rate of substitution (MRS) is the ratio of her marginal utilities. Person 1 has the utility function

$$U^1 = \ln x_1 + \ln z_1$$

So that her marginal utilities of consumption of the private good and of the public good are

$$MU^1_x = \frac{1}{x_1}$$

and

$$MU^1_z = \frac{1}{z_1}$$

respectively, and therefore her MRS is

$$MRS^1 = \frac{x_1}{z_1}$$

Since person 2's utility function is

$$U^2 = \ln x_2 + 2 \ln z_2$$

therefore

$$MU^2_x = \frac{1}{x_2}$$

$$MU^2_z = \frac{2}{z_2}$$

and

$$MRS^2 = \frac{2x_2}{z_2}$$

Since good Z is a pure public good, it is efficient to let both people consume the full amount Z, so that $z_1 = z_2 = Z$. The Samuelson condition for efficient provision is that

$$MRS^1 + MRS^2 = MRT$$

Since the equation of the production possibility curve is

$$X + Z = 120$$

the MRT, which is the negative of the slope of the production possibility curve, equals 1.

Also, since good X is a private good, $x_1 + x_2 = X$, which implies, from the equation of the production possibility frontier, that

$$x_2 = 120 - Z - x_1$$

which means that the Samuelson condition for efficiency can be written

$$\frac{x_1}{Z} + \frac{2[120 - x - 1 - Z]}{Z} = 1$$

or

$$x_1 + 2[120 - x_1 - Z] = Z$$

so that

$$x_1 + 3Z = 240 \tag{$*$}$$

defines all the efficient allocations, with x_2 defined by

$$x_2 = 120 - Z - x_1 \tag{$**$}$$

Any allocation (x_1, x_2, Z) — with x_1, x_2 and Z all non–negative — satisfying equations $(*)$ and $(**)$ will be an efficient allocation.

The efficient quantity Z of the public good actually can be anything between 60 (when $x_1 = 60$ and $x_2 = 0$) and 80 (when $x_1 = 0$ and $x_2 = 40$).

2. The two people in question #1 each have *Cobb–Douglas* preferences. From intermediate micro (AS/ECON 2300), the demand functions for goods x and z when preferences can be represented in the form

$$U(x, z) = a \ln x + b \ln z$$

are

$$x^D = \frac{a}{a+b} \frac{M}{p_x}$$

$$z^D = \frac{b}{a+b} \frac{M}{p_z}$$

where M is the person's income, x^D and z^D are the quantities demanded, and p_x and p_z are the prices of the goods.

To solve for the optimal quantity of the public good, the inverse demand functions are needed, expressing the price a person is willing to pay as a function of the person's income and the quantity she is consuming of the public good. Re–arranging the demand function above for good z, the inverse demand function is

$$p_z = \frac{b}{a+b} \frac{M}{Z}$$

Plugging in the values of a and b from the utility functions given in question #1, and the the values of M given in this question, the heights of the two people's demand curves for the public good are

$$p_z^1 = \frac{60}{2Z} = \frac{30}{Z}$$

$$p_z^2 = \frac{2}{3} \frac{60}{Z} = \frac{40}{Z}$$

Here the marginal cost of the public good is 1 (from question #1), so that the Lindahl allocation is determined by the equation

$$p_z^1 + p_z^2 = 1$$

or

$$\frac{30}{Z} + \frac{40}{Z} = 1$$

so that $Z = 70$ is the quantity of the public good at the Lindahl equilibrium, with person 1 paying taxes of 30 and person 2 taxes of 40 (and the two people facing Lindahl prices of 3/7 and 4/7 respectively).

3. If the quantity provided of the public good is Z^*, and the price person #1 must pay per unit of the public good is p_1, then she gets benefits equal to the area under her true demand curve $\phi_1(Z)$, and pays taxes of $p_1(Z)Z$. Thus her net benefits are

$$NB = \int_0^{Z*} \phi_1(Z)dZ - p_1(Z^*)Z^*$$

The first term is the area under her (true) demand curve for the public good, the benefits of public good consumption and the second term is the total taxes she must pay.

The person has some influence on the quantity Z^* and the tax rate $p_1(Z^*)$ she must say, since they are in part determined by what demand function she reports.

If she reports a slightly higher demand function, then the overall quantity Z^* of the public good will increase, since the government will choose Z^* from the rule

$$p_1(Z^*) + p_2(Z^*) + p_3(Z^*) + \cdots + p_N(Z^*) = c$$

A higher $p_1(Z)$ function increases the vertical sum of people's reported willingnesses to pay (the left side of the equation), so that the level Z^* chosen by the government will increase. Thus exaggerating one's taste for the public good (reporting $p_1(Z)$ above the person's true $\phi_1(Z)$) will increase the level of Z^* and understating the taste will decrease the government's provision of Z^*.

What about the tax price the person actually pays, $p_1^(Z^*)$? From the government's rule

$$p_1(Z^*) = c - p_2(Z^*) - p_3(Z^*) - \cdots - p_N(Z^*)$$

Demand curves slope down. Increasing Z^* must decrease each other person's $p_i(Z^*)$, and thus *increase* person 1's Lindahl price $p_1(Z^*)$. Thus exaggerating one's taste for the public good will have two effects : more of the public good and a higher price per unit that the person has to pay.

What is the change in the person's overall net benefit from exaggerating her taste? From the definition of the net benefit, the change in this net benefit would be

$$d(NB) = \phi_1(Z^*)dZ^* - p_1(Z^*)dZ^* - Z^*(dp_1)$$

If the person told the truth, and reported her actual inverse demand curve $\phi_1(Z)$, then $\phi_1(Z^*) = p_1(Z^*)$, so that

$$d(NB) = -Z^*(dp_1)$$

Now exaggerating the benefits a little would increase p_1, so that NB would *decrease.* It would not pay the person to overstate her benefits from the public good. On the other hand, understating her benefits would cause p_1 to fall, which means, from the above equation, that her net benefits would increase.

In short, it would pay the individual to understate her benefits from the public good, since by doing so she would reduce the tax price she would have to pay per unit of the public good. Understating the benefits would also have the side effect of reducing overall provision Z^* of the public good, which harms her. But at the margin, the benefits from understatement (a lower tax price) more than offset the costs (less public good consumption).

(Should she report that she gets no benefits at all from the public good, $p_1(Z) \equiv 0$? Maybe. It depends on what other people do.)

4. What is optimal for one person depends on what other people do. Consider what is optimal for family #1. Let w_i be what family i has contributed and let W be the total of everyone else's contributions, that is

$$W = \sum_{i=2}^{300} w_i$$

What is the net benefit to family #1 from contributing w_1? If $W + w_1 > 210$ (all numbers here are in thousands of dollars), then the arena gets built. Family #1 gets a benefit of v_1 from having the arena. It also pays its contribution, and gets its share of the excess contributions $W + w_1 - 210$, so that its net benefit if the arena is built is

$$v_1 - w_1 + \frac{1}{300}[W + w_1 - 210]$$

From this expression, it follows that no–one will ever contribute more than necessary. If $W + w_1 > 210$, then it would pay family #1 to contribute a little less (whatever is its valuation v_1) : as long as the reduced contribution w_1' is large enough such that $W + w_1' > 210$, reducing the contribution has not prevented the arena from being built ; all it has done is reduced the family's own payments, since it only gets back 1/300 of every extra dollar it contributes.

Now what makes sense for family #1 also makes sense for any other family.

So, if families act rationally and selfishly, there should not be any excess contributions. If the decisions to make contributions are regarded as a non–cooperative game, total contributions in any Nash equilibrium will not exceed the cost (210) of the arena.

What is a likely outcome? That is, what is a Nash equilibrium to the game? Consider any pattern of contributions, such that the total of all 300 families' contributions is exactly 210, and such that no family contributes more than its true valuation (which is one of 2, 1, or 0.1). There are lots of patterns of this form. For example, every family could contribute 210/310 times its true valuation : $100(210/310)(2) + 100(210/310)(1) + 100(210)(310)(0.1) = 210$. Each high–valuation family could contribute 1.5 and each medium–valuation family could contribute 0.6, with low–valuation families contributing nothing. Or half the high–valuation families could contribute 1.8 each, the other half could contribute 1.4 each, half the medium–valuation families could contribute 0.4 each, the other half could contribute 0.3 each, and each low–valuation family could contribute 0.05.

There are an infinite number of patterns of contribution, such that $w_i < v_i$ for each family i, and such that $\sum_{i=1}^{300} w_i = 210$. Each of them is a Nash equilibrium.

Why? Consider again family #1's decision. If it contributes even a penny less than w_1, the arena doesn't get built, since the contributions added up to exactly the cost. So giving any less than w_1 results in a loss in benefits of v_1, greater than the savings from contributing less. And it has already been argued that it would make no sense to contribute more.

There are more Nash equilibria. What if each family, for example, contributed 0.05? Then the total contributions are 15, not nearly enough to pay for the arena. Every family would get its money back. No family would be willing to increase its contribution by so much (195 more) that the arena could be built. And families have no incentive to reduce their contributions, since they're getting their money back anyway.

So there lots of other Nash equilibria for which the arena does not get built. Any pattern w_i of contributions such that $\sum_{i=1}^{300} w_i < 210$, and such that $v_j - w_j < 210 - \sum_{i=1}^{300} w_i$ will also be an equilibrium.

(Note : if instead of giving back contributions, the organization kept some of the money, if there was not enough to build the arena, or if the organization distributed the money equally among all families if the arena was not built, then all these "under–contribution" patterns would no longer be equilibria. Then the only "under–contribution" equilibrium would be for nobody to contribute anything.)

5. This question does not involve a "pivot tax", since each family's tax has been specified as exactly \$700 (if the arena is built). Here, a family's reported valuation does not affect its share of the taxes. Each family will pay 0.7 in taxes (measured in thousands of dollars) if the sum of the reported valuations is 210 or more, and nothing otherwise.

So the net benefit to a family will be

$$v_i - 0.7$$

if the sum of everyone's reported valuations is 210 or more, and is zero otherwise.

So if $v_i > 0.7$, which is true for 200 of the families, then the family wants to make sure that the arena gets built. Exaggeration of the family's valuation will not affect its taxes ; these families all have an incentive to overstate wildly how much they value the arena.

For the 100 low–valuation families, the taxes they will pay if the arena is built exceed what the arena is (truly) worth to them. Each such family has an incentive to understate wildly how much they value the arena.

If families cannot report negative valuations, then the low–valuation families are pretty limited in what they can do. They'll report valuations of 0, the medium– and high–valuation families will report very high valuations, and the arena will be built.

If negative valuations can be reported (perhaps families could claim health hazards from the air in the arena), then the low–valuation families would report very negative valuations, and the other families would report very high valuations.

Here, if $v_i > 0.7$, the family wants to report as high a value for w_i as is allowed, and if $v_i < 0.7$, the family wants to report as low a value of w_i as is allowed.