## AS/ECON 4080 ANSWERS TO ASSIGNMENT 1

1. The marginal rate of transformation ($MRT$) is the (absolute value of the) slope of the production possibility frontier. Here the production possibility frontier has the equation $X + Z = 102$, so is a straight line with slope $-1$. So $MRT = 1$. A person's marginal rate of substitution ($MRS$) of the pure public good for the pure private good is $U_z/U_x$, so that

$$MRS^1 \equiv \frac{U_Z(x_1, Z)}{U_x(x_1, Z)} = \frac{2}{Z}$$

and

$$MRS^2 \equiv \frac{U_Z(x_2, Z)}{U_x(x_2, Z)} = \frac{10}{Z}$$

(where I have used the facts that the derivative of $\ln Z$ is $1/Z$, and that $z_1 = z_2 = Z$ if the good $Z$ is non–rival, and if the allocation is efficient. Therefore, the condition that $MRS^1 + MRS^2 = MRT$ becomes

$$\frac{2}{Z} + \frac{10}{Z} = 1$$

or

$$Z = 12$$

In this case, there is a single optimal quantity of the public good to provide — $Z = 12$ — since each person's $MRS$ is independent of her consumption of the private good. The optimal allocations are any allocations $(x_1, x_2)$ such that $x_1 + x_2 = 90$, and $Z = 12$.

2. Since the production possibility frontier here is the same as in question #1, the $MRT$ here again equals 1. As in question #1, $MRS^1 = \frac{2}{Z}$. Now

$$MRS^2 \equiv \frac{U_Z(x_2, Z)}{U_x(x_2, Z)} = \frac{x_2}{Z}$$

so that the Samuelson condition for efficiency becomes

$$\frac{2}{Z} + \frac{x_2}{Z} = 1$$

or

$$Z = x_2 + 2 \tag{1}$$

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Any allocation $(x_1, x_2, Z)$ satisfying 1 and the feasibility condition $x_1+x_2+Z = 102$ will be efficient (if $x_1$, $x_2$ and $Z$ are all non–negative). Substituting from 1 for $Z$ into the feasibility constraint, $(x_1, x_2)$ will be part of an efficient allocation if

$$x_1 + 2x_2 + 2 = 102$$

or

$$x_1 + 2x_2 = 100$$

So pick any $x_1$ between 0 and 100. Then there will be an efficient allocation in which person 1 gets private good consumption of $x_1$, and in which person 2 gets private good consumption of $(100 - x_1)/2$, and in which public good consumption is $Z = \frac{100-x_1}{2} + 2 = \frac{104-x_1}{2}$. For example, $(x_1 = 0, x_2 = 50, Z = 52)$, $(x_1 = 20, x_2 = 40, Z = 42)$ and $(x_1 = 80, x_2 = 10, Z = 12)$ are all efficient allocations.

3. In this case, the height of a person's demand curve is her marginal willingness to pay for the good : how much a little more of that good is worth to her. Since $B(Q)$ is the total benefit she gets from consuming $Q$ units of the good, $B'(Q)$ is her marginal willingness to pay, the increase in the total amount that she is willing to pay in response to a small increase in the quantity $Q$ which she is consuming. The marginal cost of the good is $C'(Q)$. An allocation is efficient if the sum of all the people's willingness to pay for the non-rival good equals the marginal cost of the good, or

$$NB'(Q) = C'(Q) \tag{2}$$

The value $Q^*$ of $Q$ which satisfies equation 2 is the efficient quantity of the non-rival good to provide. (Since the curve $NB'(Q)$ slopes down, and the curve $C'(Q)$ slopes up, there is at most one value of $Q^*$ which satisfies 2.) Since the good is non-rival, it would be inefficient to exclude anyone from consuming all the quantity available. That means that the government should set the user charge $u$ low enough that each person will want to buy $Q^*$ units of the good at a unit price of $u$. [It doesn't cost anyone anything to let a person consume all the quantity available, so she should not be excluded if the quantity has been produced.] So any efficient two-part tariff should have

$$u \leq B'(Q^*)$$

(If $u < B'(Q^*)$ then each person will want to buy more of the good than is available, so that $u = B'(Q^*)$ is a nice choice for the user charge.) Since the charges must cover the cost of the good, it must be the case that

$$N(F + uQ^*) = C_0 + C(Q^*)$$

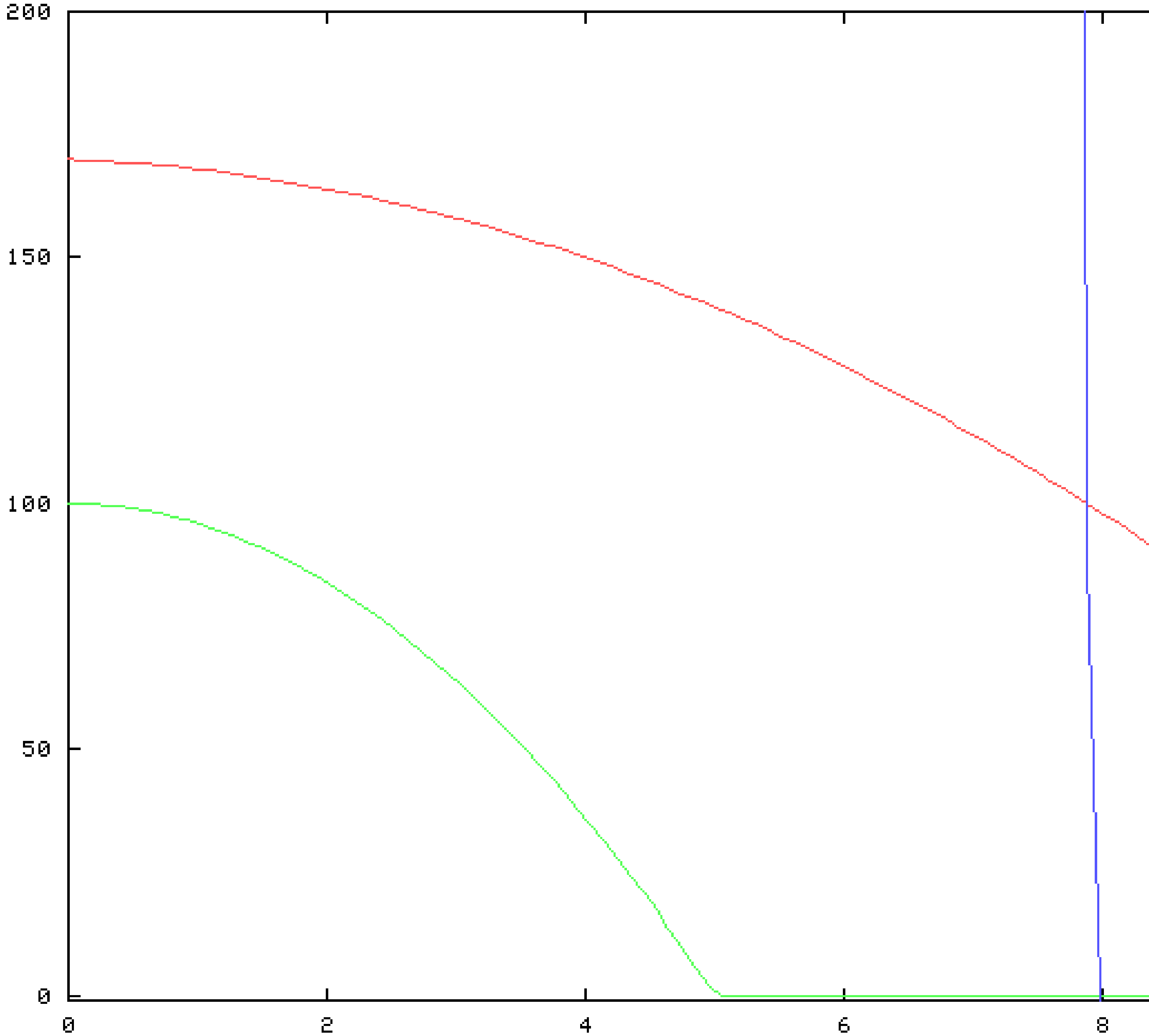
which determines the flat fee $F$ as

$$F = \frac{C_0 + C(Q^*)}{N} - uQ^*$$

If the fixed cost $C_0$ is small enough, then the flat fee might be negative : charging user charges equal to people's marginal willingness ot pay might more than cover costs. If the fixed cost were large enough, then it might be a good idea not to provide the non-rival good at all : the benefit from providing the good exceed the costs if and only if

$$NB(Q^*) \geq C_0 + C(Q^*)$$

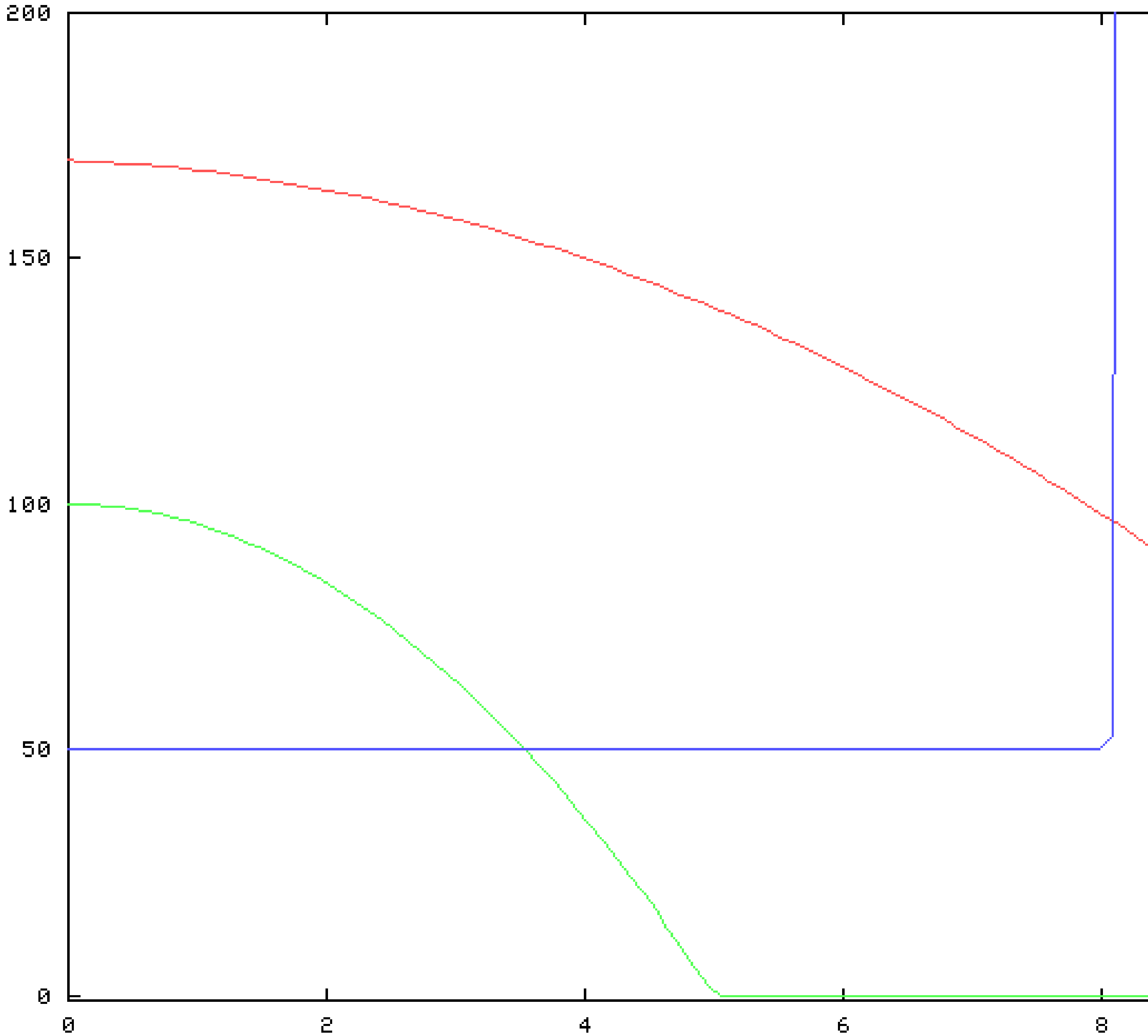
4. A person's marginal willingness to pay for trips outside of rush hour is $B_0'(t)$, which defines a downward-sloping inverse demand curve for bus trips outside of rush hour. But the marginal willingness to pay $B_0'(t)$ hits zero at $\bar{t}$, which is less than $T/N$. So if trips on the bus oustide of rush hour were free, people would only demand $\bar{t}$trips each, for a total demand of $N\bar{t} < T$. So bus trips are a non-rival good outside of rush hour : even if they were free, there would be seats available. This condiiton implies that the efficient fare $f_0$ to charge outside of rush hour is 0. Charging any fare higher than zero would exclude people from the benefits of consuming a non-rival good, which is an inefficient practice. Since the capacity of the bus sytem is fixed in this question, and since the cost of running the bus system do not vary with the number of passengers, then nobody is harmed by allowing one more person (or one more trip) outside of rush hour. Any positive fare would exclude people who derive positive benefits from another trip.




Matters are different during rush hour. Since $B_r'(\frac{T}{N}) > 0$, then there would be excess demand during rush hour if trips were free. During rush hour, trips are rival. Efficiency requires that the fare during rush hour be set so that the total number of trips demanded $Nt$ equal the available capacity $T$. That means setting $f_r = B_r'(\frac{T}{N})$. A higher fare than that would mean that the bus system was not being used fully in rush hour. A lower fare than that would mean there would be excess demand. Excess demand would mean some inefficient mthod of rationing the available spaces : people might have to arrive twenty minutes early for the bus, and just wait around, in order to be assured of gtting a seat. Given that $f_0 = 0$, and that $f_r = B_r'(\frac{T}{N})$, the head tax per person will be set to cover costs, so that $hN + f_r T = C$, or

$$h = \frac{C}{N} - B_r'(\frac{T}{N})\frac{T}{N}$$

5. Now there is a cost to one more trip being taken outside of rush hour, even though there is no congestion on the bus. Each additonal trip taken adds $c$ to the total operating costs. So the transit authority should set fares outside of rush hour so as to cover those marginal operating costs : $f_0 = c$.




What fares should be charged during rush hour depends on the relation between $c$ and $B_r'(\frac{T}{N})$. Certainly the transit authority should not set rush-hour fares below $c$, since then people would be valuing trips at the margin at less than the added cost of providing the trip. If $c > B_r'(\frac{T}{N})$, then the fare should equal $c$ during rush hour (as well as out of rush hour). The buses would be uncongested all the time, since at the margin, people are unwilling to pay the added operating cost of one more trip, at any time of day. If $c < B_r'(\frac{T}{N})$, then the authority should set the rush hour fare at $B_r'(\frac{T}{N})$, just as in question #4. Raising the fare above $B_r'(\frac{T}{N})$ would mean total demand for seats during rush hour would be less than available capacity. This would be inefficient : people would be excluded from taking trips, even thought there were seats available, and even though they were willing to pay more than the added operating costs $c$ of the trip. In short, in this case

$$f_0 = c \ \ f_r = \max(c, B_r'(\frac{T}{N}))$$

As in question #4, the tax $h$ can be set to make sure that all costs are covered. If $f_r = c$, then $h = C/N$, since fares are covering variable costs. If $f_r = B_r'(\frac{T}{N})$, then

$$h = \frac{C}{N} - B_r'(\frac{T}{N})\frac{T}{N}$$

as in question #4.