AS/ECON 4080 Answers to Assignment 2 March 2004

1. Here are some axioms that the rule **obeys** :

It obeys the Pareto principle (#3 in the text) : if every person preferred A to B, that means 100% prefer A to B, which is certainly more than 80%; so whenever A is preferred unanimously to B, it will be ranked higher than B.

It is non–dictatorial (#6 in the text) : if one person prefers A to B, and everyone else prefers B to A, then A will **not** be ranked above A.

Independent alternatives are irrelevant (#5 in the text) : the ranking of A and B depends only on how people rank those alternatives relative to each other, not on how they rank either of them compared to some third alternative C.

But it certainly violates transitivity (#4 in the text), for example if people's preferences over 3 alternatives a, b and c are as described in the table below :

1	2	3
a	a	b
b	c	a
c	b	c

Since only two–thirds of the people prefer a to b, a and b are ranked as tied. Since only two–thirds of the people prefer b to c, b and c are ranked as tied. But 100% of the people rank a above c, so that a and c are not tied, which means that the social ranking is not transitive.

It also does not always produce a top–ranked alternative (that is, #1 in the text can be violated). In the following example, a is ranked above b which is ranked above c which is ranked above d which is ranked above e which is ranked above f, but f is ranked above a. (This example also shows that the rule is not transitive.)

1	2	3	4	5	6
a	b	c	d	e	f
b	c	d	e	f	a
c	d	e	f	a	b
d	e	f	a	b	c
e	f	a	b	c	d
f	a	b	c	d	e

2. Voter #1 is the voter who likes alternative a the most. So she would like to find a rule that guarantees a win for a.

So first she should find out what alternatives a can beat in a two–way vote. From the table, a can beat b in a two–way vote (voters #1 and #3 prefer a to b), and it can defeat d (voters #1 and #2 prefer a to d). But it would lose to c in a two–way vote, since both voters #2 and #3 prefer c to a.

Voter #1, when organizing the votes, wants to make sure that a does not get matched up against c in the first stage, since a would lose. So she should match a against b or d in the first stage.

But she also wants a to win the second stage, which means that she has to ensure that c does not win the first stage (because it would then beat a in the second stage). So she has to find an alternative that two out of three voters prefer to c. From the table, that's b : voters #1 and #2 will both vote for b over c. (But d will lose to c in any vote.)

What she should do is match a against d, and b against c, in the first stage. a and b would win those first votes, and then a would defeat b in the second stage. That way is the only way of getting a to win both votes, if people vote sincerely.

But they won't want to vote sincerely. Consider person #2, who ranks a fairly low. In the first stage, he would vote for b over c, and a over d if he voted sincerely, only to see b lose to a. If others voted sincerely, he would be better voting strategically, voting for d over a in the first stage, even though he actually ranks a higher than d.

Why should he vote for d in the first stage? That would make d win the first stage (since if voter 3 votes sincerely, she votes for d over a). Asuming that everyone voted sincerely in the other first stage election (so that b beat c), this strategic behaviour would set up a second–stage vote between b and d, which would be won by b, voter #2's overall favourite.

Therefore, strategic voting leads to a more attractive outcome, from voter #2's point of view : sincere voting leads to an outcome of a, and strategic voting leads to an outcome of b, his most preferred outcome.

3. If a voter has income of 24, and expects to pay for one–third of the cost of highways and education, then her private food exenditure x_i would equal $24 - (H + E)/3$. So her overall utility if a policy (H, E) were implemented would be

$$24 - \frac{H + E}{3} + a_i \ln H + b_i \ln E$$

To find the policy she would like best, take the derivatives with respect to H, and with respect to E, and set them equal to 0.

The derivative with respect to H is

$$-\frac{1}{3} + \frac{a_i}{H}$$

which equals zero when $H = 3a_i$. The derivative with respect to E is

$$-\frac{1}{3} + \frac{b_i}{E_i}$$

which equals zero when $E = 3b_i$.

Therefore, each voter's preferred policy is $(3a_i, 3b_i)$. So voter #1 wants $(18, 18)$, voter #2 wants $(9, 6)$ and voter #3 wants $(3, 27)$.

4. Suppose that highway expenditure H varies, holding constant education expenditure E. If voter i pays for 1/3 of the cost of the highway expenditure, then her utility is $24 - (H+E)/3 + a_i \ln H + b_i \ln E$. Question #3 showed that the derivative of this utility with respect to H is

$$\frac{a_i}{H} - \frac{1}{3}$$

This derivative is positive whenever $H < 3a_i$, and is negative whenever $H > 3a_i$. Therefore, her preferences are single–peaked, with a unique local maximum at $H = 3a_i$. Similarly, the derivative with respect to E is $b_i/E - 1/3$, which is positive when $E < 3b_i$ and negative when $E > 3b_i$, and so is single–peaked.

If people vote over highway expenditure, the single–peaked preferences imply that the median of the preferred levels $3a_i$ of highway expenditure will win, namely $H = 9$. If they vote over education expenditure, then the median preferred level of $3b_i$, $E = 18$, will win.

So the likely outcome is $(H, E) = (9, 18)$. Any proposal to change H from 9, or to change E from 18, will lose. (For example, voters 1 and 2 would vote against any proposal to lower H below 9, and voters 2 and 3 would vote against any proposal to increase highway expenditure above 9.)

5. Let T be total expenditure, and let h be the fraction of total expenditure spent on highways, so that $H = hT$ and $E = (1-h)T$. Now $x_i = 24 - T/3$, and a voter's utility can be written

$$24 - \frac{T}{3} + a_i \ln hT + b_i \ln (1-h)T$$

The derivative of this utility with respect to T is

$$-\frac{1}{3} + \frac{a_i}{T} + \frac{b_i}{T}$$

This expression is positive whenever $T < 3(a_i + b_i)$, and negative whenever $T > 3(a_i + b_i)$. Therefore, in voting over total expenditure T (taking as given the share h going to highways), each voter has single–peaked preferences, with a peak at $3(a_i + b_i)$. The median voter theorem applies, and the winning level of total expenditure T is the median of the preferred levels $3(a_i + b_i)$, in this case $T = 30$. Any increase in T above 30 would be opposed by a coalition of #2 and #3 ; any proposal to decrease T below 30 would be defeated by a coalition of #1 and #3.

The derivative of utility with respect to the share h of expenditure going to highways is

$$\frac{a_i}{h} - \frac{b_i}{1-h}$$

This derivative is positive whenever $h < a_i/(a_i + b_i)$,and negative whenever $h > a_i/(a_i + b_i)$. Thus preferences for each voter, over the share of expenditure going to highways, are single–peaked, with a peak at $h = a_i/(a_i + b_i)$. The median voter theorem applies : here $a_i/(a_i + b_i)$ equals 1/2, 3/5 and 1/10 for the three voters. The median level is 1/2. Therefore, spending exactly half the budget on highways will defeat any other proposed share, $h = 1/2$ is the winner.

Therefore, in this case, the likely outcome is $T = 30$ and $h = 1/2$, meaning that $H = 15$ and $E = 15$. Notice that this is a different outcome than would occur if voters voted on H and E separately, as in question #4.