AS/ECON 4080 Answers to Assignment 1 February 2005

1. If good Z is non–rival, and good X is rival, then in **any** efficient allocation, the Samuelson condition must hold :

$$MRS^1_{ZX} + MRS^2_{ZX} + \cdots + MRS^N_{ZX} = MRT$$

if there were N people.

Any efficient allocation must also have aggregate production which is efficient : that is (Z, X) must be on the production possibility frontier, if there were only two goods, and if Z and X denoted the total quantities produced of the goods.

If good Z is a pure public good (and not a pure public bad), then it will be efficient to have

$$z_1 = z_2 = \cdots = z_N = Z$$

With only two goods, one rival and the other non–rival, these two conditions are the only requirements for efficiency. In this question, in which there are only two people, then an efficient allocation is any list (x_1, x_2, Z) of quantities of goods allocated to the two people, such that each quantity is non–negative, such that the list satisfies the Samuelson condition, and such that the aggregate production $(x_1 + x_2, Z)$ is efficient.

In this case, since the production possibility frontier has the equation

$$X^2 + Z^2 = 180$$

then it also can be written

$$X = \sqrt{180 - Z^2} \equiv F(Z) \tag{1}$$

so that the marginal rate of transformation, $-F'(X)$, is, from differentiation of equation (1) above,

$$MRT = -F'(Z) = \frac{Z}{\sqrt{180 - Z^2}} = \frac{Z}{X} \tag{2}$$

The Samuelson condition is

$$MRS^1_{ZX} + MRS^2_{ZX} = MRT$$

Since

$$U^1(x_1, Z) = \ln x_1 + 4 \ln Z$$

then

$$\frac{\partial U^1}{\partial x_1} = \frac{1}{x_1}$$

$$\frac{\partial U^1}{\partial Z} = \frac{4}{Z}$$

so that

$$MRS^1 = \frac{4x_1}{Z} \tag{3}$$

Similarly

$$MRS^2 = \frac{4x_2}{Z} \tag{4}$$

That means that in this question, equations (2), (3) and (4) imply that the Samuelson condition is

$$\frac{4x_1}{Z} + \frac{4x_2}{Z} = \frac{Z}{X} \tag{5}$$

Since $x_1 + x_2 = X$, equation (5) can be written

$$\frac{4X}{Z} = \frac{Z}{X} \tag{6}$$

which is the same as

$$4X^2 = Z^2$$

or

$$Z = 2X \tag{7}$$

Substituting equation (7) into the equation of the production possibility frontier,

$$5X^2 = 180$$

so that $X = 6$, and $Z = 12$.

In this case (because the two people have identical, homothetic preferences), there is a unique efficient level of public good production, $Z = 12$. Any allocation (x_1, x_2, Z), with $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 = 6$, and $Z = 12$ will be efficient.

2. In this question, the equation of the production possibility frontier,

$$X + Z = 36$$

can be written

$$X = 36 - Z \equiv F(Z) \tag{1}$$

so that the marginal rate of transformation is

$$MRT \equiv -F'(Z) = 1 \tag{2}$$

Since person 1's utility function is

$$U^1 = 2 \ln x_1 + 3 \ln Z$$

then

$$\frac{\partial U^1}{\partial x_1} = \frac{2}{x_1}$$

$$\frac{\partial U^1}{\partial Z} = \frac{3}{Z}$$

so that

$$MRS^1_{ZX} = \frac{3x_1}{2Z} \tag{3}$$

And

$$\frac{\partial U^2}{\partial x_2} = \frac{2}{x_2}$$

$$\frac{\partial U^2}{\partial Z} = \frac{1}{Z}$$

so that

$$MRS^2_{ZX} = \frac{x_2}{2Z} \tag{4}$$

Now an efficient allocation is any non–negative (x_1, x_2, Z) with

$$x_1 + x_2 + Z = 36 \tag{5}$$

and with

$$\frac{3x_1}{2Z} + \frac{x_2}{2Z} = 1 \tag{6}$$

The Samuelson condition (6) can be written

$$3x_1 + x_2 = 2Z \tag{7}$$

Any allocation (x_1, x_2, Z) that satisfies equations (5) and (7),with (x_1, x_2, Z) all non–negative, will be efficient. Substituting for $x_2 = 36 - x_1 - Z$ from equation (5) into equation (7) reduces these two equations to

$$3x_1 + (36 - x_1 - Z) = 2Z$$

or

$$2x_1 = 3Z - 36 \tag{8}$$

For example, $x_1 = 0$, $x_2 = 24$, $Z = 12$ satisfies equations (5) and (7), and is an efficient allocation. So are $x_1 = 6$, $x_2 = 14$, $Z = 16$, and $x_1 = 12$, $x_2 = 4$, $Z = 20$, and $x_1 = 14.4$, $x_2 = 0$, $Z = 21.6$.

The general form for all efficient allocations in this example is, then : x_1 can be any quantity between 0 and 14.4 ; whatever x_1 is, then $Z = \frac{2}{3}x_1 + 12$ (from equation (8)), and $x_2 = 24 - \frac{5}{3}x_1$.

3. To answer this question, what is needed first are the demand functions when a person has *Cobb–Douglas* preferences, a topic covered, for example, in the appendix to chapter 5 of Varian's textbook.

If

$$U(x, z) = (1 - \alpha)\ln x + \alpha \ln z$$

then

$$\frac{\partial U}{\partial x} = \frac{(1 - \alpha)}{x}$$

$$\frac{\partial U}{\partial z} = \frac{\alpha}{z}$$

so that

$$MRS_{zx} = \frac{\alpha x}{(1 - \alpha) z} \tag{1}$$

If the person has income Y, and can buy the two goods at prices p_x and p_z respectively, then she will choose a consumption bundle where her indifference curve is tangent to her budget line. This tangency means that

$$MRS_{zx} = \frac{p_z}{p_x}$$

or (from equation (1))

$$\frac{\alpha x}{(1 - \alpha) z} = \frac{p_z}{p_x} \tag{2}$$

in this case. The fact that she is on her budget line means that

$$p_x x + p_z z = Y \tag{3}$$

To find the person's demand function for good z, equations (2) and (3) must be solved to get z as a function of Y, p_x and p_z.

Equation (2) can be written

$$\alpha p_x x = (1 - \alpha) p_z z \tag{4}$$

Substituting from (4) for $p_x x$ in (3) yields

$$\frac{(1 - \alpha)}{\alpha} p_z z + p_z z = Y$$

or

$$\frac{(1 - \alpha)}{\alpha} p_z z + \frac{\alpha}{\alpha} p_z z = Y$$

meaning that

$$p_z z = \alpha Y \tag{5}$$

so that the demand curve for the public good of person 1 is

$$Z_1^D = \alpha \frac{Y_1}{p_z^1} \tag{6}$$

if she had income of Y_1,and could buy the public good at a price of p_z^1.

For solving the Lindahl equilibrium, we need the inverse demand function, the price as a quantity consumed, which is

$$p_z^1 = \alpha \frac{Y_1}{Z} \tag{7}$$

Similarly, person 2's inverse demand function for the public good is

$$p_z^2 = \beta \frac{Y_2}{Z} \tag{8}$$

The Lindahl pricing rule is to set the sum of the two people's Lindahl prices equal to the MRT, or

$$\alpha \frac{Y_1}{Z} + \beta \frac{Y_2}{Z} = MRT$$

which can be written

$$\alpha Y_1 + \beta Y_2 = (MRT)Z \tag{9}$$

If the MRT is a constant, then

$$Z = \frac{\alpha Y_1 + \beta Y_2}{MRT} \tag{10}$$

so that increasing Y_2 by \$1000 and decreasing Y_1 by \$1000 would increase the optimal public good supply by

$$(\beta - \alpha)\frac{1000}{MRT}$$

If $\beta > \alpha$, then person 2 has a stronger taste for the public good than person 1. In particular, his income elasticity of demand for the public good would be higher than person 1's. So transferring money from person 1 to person 2 would increase the overall quantity of the public good, when the Lindahl pricing rule is used.

4. How many trips to the park would a person take, if she had to pay an admission charge of p per trip? If she took m trips, at a price of p each, then she would have $y - T - pm$ left to spend on the private good, if her income were y and if she had to pay a head tax of T. That means that her overall utility, if she faced a price of p, per trip, and chose to take m trips, would be

$$y - T - pm + [12m - m^2]Z \tag{1}$$

The number of trips that she actually would choose to take is the value m which maximizes her utility (1). Taking the derivative of expression (1) with respect to m, and setting it equal to zero, yields

$$(12 - 2m)Z - p = 0 \tag{2}$$

or

$$m = \frac{12Z - p}{2Z} \tag{3}$$

which is her demand function for trips to the park, as a function of the size of the park, and of the price p charged per visit.

Now if I substitute for m back into equation (1), her overall utility, if she had to pay a price p for each trip, and if she chose her preferred number of trips, would be

$$y - T - p\frac{12Z - p}{2Z} + [12\frac{12Z - p}{2Z} - (\frac{12Z - p}{2Z})^2]Z$$

or

$$y - T + \frac{12Z - p}{2Z}[12Z - \frac{12Z - p}{2Z}Z - p]$$

which equals

$$y - T + \frac{1}{4Z}(12Z - p)^2 \tag{4}$$

Now if the government builds a park of size Z, and charges an admission price of p per visit, then each person will make $(12Z - p)/2Z$ visits (from equation (3)), so that the total revenue it will collect from admission charges is

$$\frac{12Z - p}{2Z}Np \tag{5}$$

if there are N people in the district. The cost of the park is Z^2, so that the government must raise the remaining revenue,

$$Z^2 - \frac{12Z - p}{2Z}Np \tag{6}$$

from head taxes. Since there are N people paying the same head tax, the head tax is expression (6) divided by N, or

$$T = \frac{Z^2}{N} - \frac{12Z - p}{2Z}p \tag{7}$$

Substituting from equation (7) into expression (4), the utility of a typical citizen will be

$$y - \frac{Z^2}{N} + \frac{12Z - p}{2Z}p + \frac{1}{4Z}(12Z - p)^2 \tag{8}$$

if the government builds a park of size Z, and charges an admission fee of p per visit.

This expression is the same as

$$y - \frac{Z^2}{N} + (12Z - p)[\frac{12Z - p}{4Z} + \frac{p}{2Z}]$$

or

$$y - \frac{Z^2}{N} + \frac{(12Z - p)(12Z + p)}{4Z}$$

which can be written

$$y - \frac{Z^2}{N} + \frac{144Z^2 - p^2}{4Z} \tag{9}$$

The optimal size of park and admission fee are the values of Z and p which makes each citizen as well–off as possible : so we maximize expression (9) with respect to p and Z. Taking the partial derivatives of (9) with respect to p and Z respectively, and setting them equal to zero, yields

$$\frac{-2p}{4Z} = 0 \tag{10}$$

$$36 + \frac{p^2}{4Z^2} - \frac{2Z}{N} = 0 \tag{11}$$

Equation (10) says that the optimal charge per visit should be zero! Substituting $p = 0$ into equation (11), the optimal size Z for the park is

$$Z = 18N \tag{12}$$

Equation (12) is actually the Samuelson condition. If the price is set optimally, at $p = 0$, then equation (3) says that each person takes $m = 6$ trips to the park. Then the person's utility can be written

$$U = x + (12(6) - 6^2)Z = x + 36Z \tag{13}$$

Equation (13) implies that the person's MRS, U_Z/U_x equals 36. Since the cost of the park is Z^2, then

$$MRT = 2Z$$

so that the Samuelson condition, when there are N people in the district, is

$$N(MRS) = 36N = MRT = 2Z \tag{14}$$

which is just equation (12).

The most important point is the optimality of a price of \$0 per trip. And that result can be derived without much of the mathematics. Suppose that a price $p > 0$ per trip is charged, as in the accompanying figure.

Charging a positive price yields tax revenue, as labelled in the figure ((1200)(3) = 3600 dollars per person in the figure). Lowering the price to 0 would lose deprive the government of this revenue, forcing them to raise the head tax by the amount of the lost revenue.

But lowering the price would also benefit the each citizen, by inducing her to take more trips to the park. The measure of the consumer's benefit is the area under her demand curve. Lowering the price increases each consumer's benefit (her consumer surplus) by the area beneath the demand curve, between the heights of 0 and p. In the figure, that gain is the rectangle labelled "tax revenue", plus the triangle labelled "excess burden". (Its value is (1200)(3) + (0.5)(3)(1200) = 5400 dollars in the figure.)

The triangle is labelled "excess burden", because it is the excess burden, or deadweight loss, or charging a price in excess of marginal cost (the marginal cost of allowing another trip is zero here). Lowering the price to zero, and raising the head tax to make up for the lost revenue, gives net gain to each citizen of the area of that triangle. So — regardless of what is the size Z of the park — it always makes citizens better off to lower the price of admission if the price is positive. The optimal price is 0, if any revenue shortfall can be covered by a head tax.

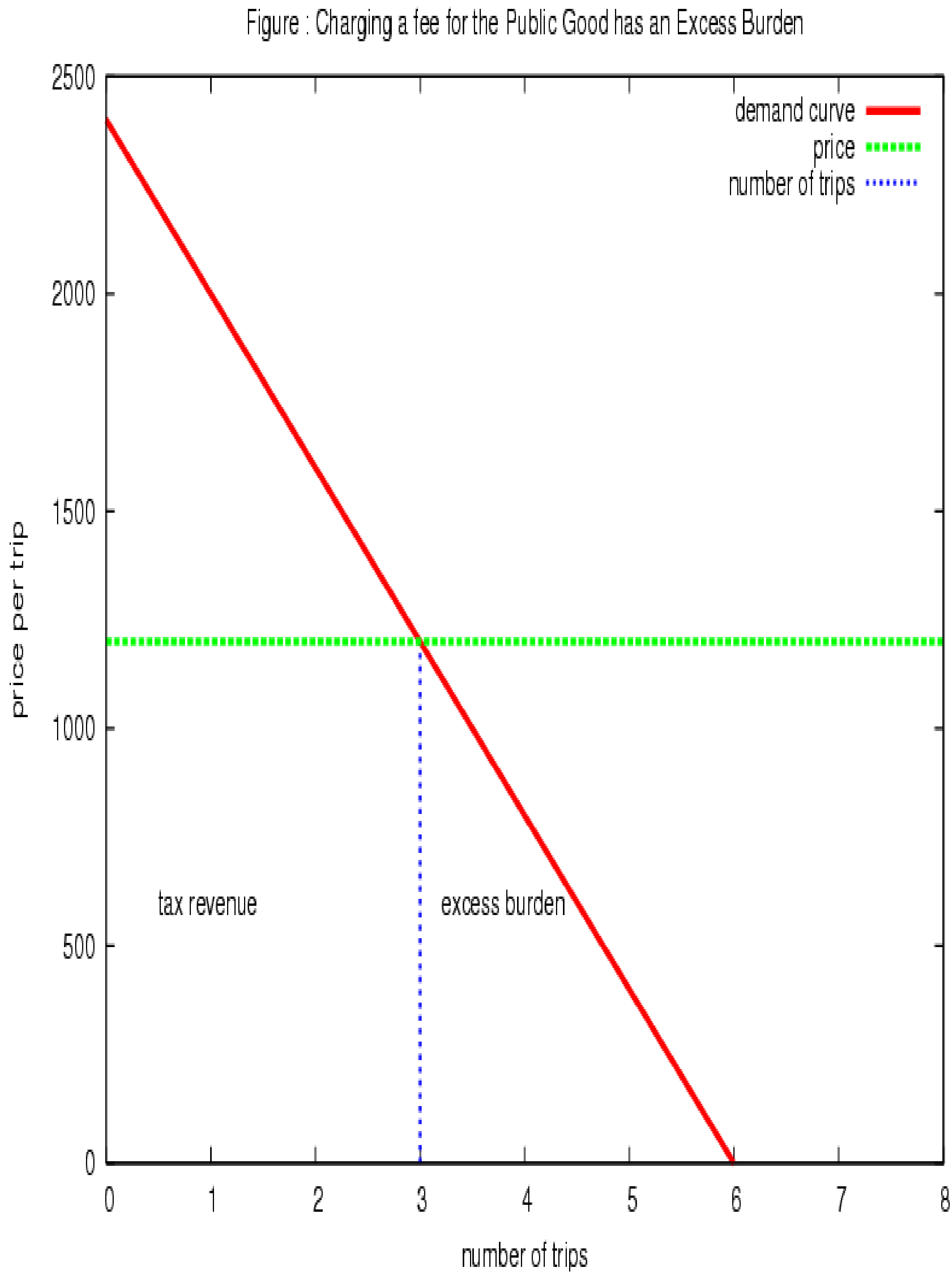
Figure : Charging a fee for the Public Good has an Excess Burden
demand curve
price
number of trips
price per trip
number of trips
tax revenue
excess burden
2500
2000
1500
1000
500
0
0
1
2
3
4
5
6
7
8

5. What is the net benefit — to everyone — of having some driver take one more trip? The net benefit to the driver herself is MB. Suppose that there are N other drivers on the road at the same time. Each of these other drivers may be affected, by having the first driver take this additional trip. For each of these N drivers, the additional trip increases the number O of other drivers on the road, and so affects each of these N drivers by $\partial MB/\partial O$.

So if the road is not congested, that is if the number of drivers on the road is less than $\bar{O}$, having the driver take one more trip imposes no costs on other drivers. If the driver wants to take the trip, she should be allowed to do so. In other words, if the number of drivers on the road is less than $\bar{O}$, then an additional trip should be encouraged if the driver gets a positive net benefit MB from it. That means that the toll should be 0, if the number of drivers on the road at that particular time of day would be $\bar{O}$ or less (in the absence of tolls). Charging a toll would discourage people from taking trips, which do not harm other drivers, and which would yield positive benefits to the driver taking the trip.

But if the road is congested, then each additional trip imposes costs of

$$-\frac{\partial MB}{\partial O}$$

on each of the N drivers already on the road. The overall net benefit — to all drivers together — is

$$MB + N\frac{\partial MB}{\partial O} \tag{1}$$

The first term in the above expression is the benefit to the driver taking the trip ; the second term is the costs imposed on the N other drivers.

So should the driver take the additional trip? Only if the benefit to the driver, MB, exceeds the damage $-N(\partial MB/\partial O)$ that the trip imposes on other drivers.

Of course we probably don't know the exact value of each trip to each driver. How do we encourage only trips that are valued more, that is only encourage trips for which the marginal benefit to the driver is $-N(\partial MB/\partial O)$ or more?

What if we charged a toll of t per trip? Each driver would only take a trip if the value to her of the trip exceeded the toll, that is, if $MB > t$. So charging a toll of t guarantees that only trips which drivers think are important will get taken, only trips for which the marginal benefit is t or more. By setting a toll of

$$t = -N\frac{\partial MB}{\partial O} \tag{2}$$

then we guarantee that the only trips which are taken are those trips for which the benefit to the driver exceeds the costs imposed on the other driver.

Thus equation (2) defines the optimal toll for the highway. Note that the first case, a zero toll when the road is not congested, is a special case of formula (2), applicable when $O < \bar{O}$.