*Q*1. Suppose that a person's preferences over private consumption x, and the level of public expenditure z could be represented by the utility function

$$u(x,z) = 1000 - \frac{1}{x} - \frac{a}{z}$$

where $a > 0$ was some positive constant.

If the person's before–tax income were y, if public expenditure were to be financed by a proportional income tax, and if this person's share of the total income was s, how would she rank the different possible levels of public expenditure, knowing that they will be financed by the income tax?

Are her preferences over public expenditures single–peaked? Explain briefly.

*A*1. The answer is "yes" : her preferences over public expenditure are single–peaked.

One way of checking would be to look at the indifference curves. The slope of an indifference curve (when we have x on the vertical axis, and z on the horizontal) is $-MU_z/MU_x$. Here

$$MU_x = \frac{1}{x^2}$$

$$MU_z = \frac{a}{z^2}$$

so that the slope of an indifference curve is

$$-MRS = -\frac{ax^2}{z^2}$$

As we move down and to the right, z increases and x decreases, so that the slope decreases in absolute value : the indifference curves get less steep.

Whenever the person's indifference curves have this "usual" shape, then her preferences over public expenditure will be single–peaked if she pays a fixed fraction of the cost.

Alternatively, the person's utility could be computed directly. Since public expenditure is financed by an income tax, the income tax rate t is determined by

$$tY = z$$

where Y is the total income in the jurisdiction. This person's taxes are equal to s times the total taxes paid by everyone, since s is her share in total income. So if the total public expenditure is z, the taxes she must pay are sz. That means that her disposable income available for private consumption will be $y - sz$, if a level of public expenditure chosen is z. So, substituting $x = sz$, her utility will be

$$1000 - \frac{1}{y - sz} - \frac{a}{z} \qquad (1-1)$$

if public expenditure is z. How does that utility vary with z? The derivative of expression $(1-1)$ with respect to z is

$$-\frac{s}{(y-sz)^2}+\frac{a}{z^2} \qquad (1-2)$$

How does that derivative vary with z? Take the derivative once more. The derivative of expression $(1-2)$ with respect to z is

$$-\frac{2s^2}{(y-sz)^3}-\frac{2a}{z^3} \qquad (1-3)$$

Expression $(1-3)$ is negative. That means that expression $(1-2)$ is positive first, then equals zero, and then is negative. As public expenditure z increases, first her utility increases, and then her utility decreases. So her preferences are single–peaked.

*Q*2. For the person described in question #1, what is her most–preferred level of public expenditure? How does it vary with the parameter a in her utility function?

*A*2. Her most–preferred level of public expediture is the quantity for which her indifference curve is tangent to a "budget line", the slope of which is the "price" she must pay for public expenditure.

From the answer to question #1, that price is s : each dollar increase in public expenditure raises her total taxes by s. The answer to question #1 also indicates that her MRS equals $\frac{ax^2}{z^2}$, so that her most–preferred level of public expenditure is the one for which

$$\frac{ax^2}{z^2}=s \qquad (2-1)$$

Since $x=y-sz$, equaion $(2-1)$ becomes

$$\frac{a(y-sz)^2}{z^2}=s \qquad (2-2)$$

or

$$\frac{y-sz}{z}=\sqrt{s/a} \qquad (2-3)$$

so that

$$z=\frac{\sqrt{a}y}{s\sqrt{a}+\sqrt{s}} \qquad (2-4)$$

Equation $(2-4)$ can also be obtained by setting expression $(1-2)$, which measures how her overall utility changes with public expenditure, equal to 0.

Note that quantity demanded of public expenditure increases with the person's income y, and decreases with the "price" s she pays for public expenditure.

The derivative of the right side of equation $(2-4)$ with respect to a is

$$\frac{\sqrt{s}y}{2\sqrt{a}(s\sqrt{a}+\sqrt{s})^2} \qquad (2-5)$$

which must be positive. The marginal utility of public expenditure increases with the parameter a, so that a higher value of a will lead to a higher preferred level of public expenditure, other things equal.

*Q*3. If there were many voters, each with the preferences described in question #1, and all with the same preference parameter a, how would voters' preferred levels of public expenditure z vary with their income level y, if a person's share s of the taxes was proportional to her income y?

*A*3. Suppose that $s = \alpha y$ for some positive constant α. Then equation $(2-4)$ becomes

$$z = \frac{\sqrt{a}y}{\alpha\sqrt{a}y + \sqrt{\alpha}\sqrt{y}} \tag{3-1}$$

or

$$z = \frac{\sqrt{a}}{\alpha\sqrt{a} + \sqrt{\alpha}y^{-1/2}} \tag{3-2}$$

which means that z increases with income : the "direct" income effect (public expenditure is a normal good for this person) is stronger than the indirect "price" effect (higher income leads to a higher share of the tax burden). In particular, here

$$\frac{\partial z}{\partial y} = y^{3/2}\frac{\sqrt{\alpha}}{2\sqrt{a}}z^2 > 0 \tag{3-3}$$

*Q*4. Three voters are deciding on public expenditure on police services and on education. They must each pay an equal share (one third) of the cost of each public expenditure category.

Each person has the same income, 600. (All variables are measured in hundreds of dollars).

If x_i is person i's expenditure on private consumption, Y is expenditure on police services, and Z is expenditure on education, then the three people have preferences represented by the following utility functions :

$$u^1(x_1, Y, Z) = x_1 + 4\ln Y + 1\ln Z$$
$$u^2(x_2, Y, Z) = x_2 + 2\ln Y + 4\ln Z$$
$$u^3(x_3, Y, Z) = x_3 + 6\ln Y + 6\ln Z$$

What levels of expenditure would be chosen, under pairwise majority rule, if they were required to vote separately on each expenditure category, choosing police expenditure using pairwise majority rule in one vote, and choosing education expenditure in another vote?

*A*4. Suppose that a person's preferences could be represented by the utility function

$$u(x, Y, Z) = x + a\ln Y + b\ln Z$$

that she had income of 600, and that she had to pay one–third of the cost of Y and Z. Note that this description fits all 3 people : they differ only in their values of a and b.

Then this person's overall utility would equal

$$U = 600 - \frac{X+Y}{3} + a \ln Y + b \ln Z \tag{4 − 1}$$

when she takes into account that she has to pay her share of the cost of public expenditure.

The derivatives of her overall utility U with respect to Y and Z are then

$$U_Y = \frac{a}{Y} - \frac{1}{3} \tag{4 − 2}$$

$$U_Z = \frac{b}{Z} - \frac{1}{3} \tag{4 − 3}$$

respectively.

So her preferences in each dimension — Y and Z — are single–peaked, since U_Y is a decreasing function of Y, and U_Z is a decreasing function of Z. Her most–preferred level of expenditure on each category, the levels of Y and Z for which expressions $(4-2)$ and $(4-3)$ equal 0, are

$$Y^* = 3a \tag{4 − 4}$$

$$Z^* = 3b \tag{4 − 5}$$

If people voted over Y — holding education expenditure Z constant, and knowing that increases in police service expenditure would lead to higher taxes — then the median voter theorem applies. The median of the preferred levels $3a$ of police expenditure will win any pairwise vote — holding Z constant. In this question $a_1 = 4$, $a_2 = 2$ and $a_3 = 6$, so that the median of the preferred levels of police expenditure is $3(4) = 12$.

Similarly, all 3 voters have single–peaked preferences over education expenditure Z, if police expenditure is held constant. The median voter theorem applies, and the level of Z chosen is the median of $\{3b_1, 3b_2, 3b_3\}$. Since $b_1 = 1$, $b_2 = 4$, $b_3 = 6$, that median level is $(3)(4) = 12$.

So here the levels of Y and Z chosen, if people vote on them separately, are $Y = 12$ and $Z = 12$.

*Q*5. If the people have the same preferences and incomes as in the previous question #4, and if they each must pay for one–third of any public expenditure, what level of expenditure on police services and on education would they choose, if they used pairwise majority rule, but voted separately on two issues : the **aggregate amount** of public expenditure (on police and education) together in one vote, and what share of total expenditure should go to education in a separate vote?

*A*5. Now people don't get to vote on Y, holding Z constant, or on Z, holding Y cosntant.

Instead, they vote on the total expenditure $E = Y + X$, holding constant the share σ going to police expenditure, or on the share $\sigma = \frac{Y}{Y+Z}$ of expenditure devoted to police services, holding constant the total level E of expenditure.

So

$$Y = \sigma E \tag{5 − 1}$$

$$Z = (1 - \sigma)E \tag{5 − 2}$$

and the overall utility of a person (with income 600, paying for one–third of expenditures, and with preferences represented by the utility funtion $x + a \ln Y + b \ln Z$) would be

$$U = 600 - \frac{E}{3} + a \ln(\sigma E) + b \ln((1 - \sigma)E) \tag{5 − 3}$$

What happens when E changes (holding σ constant), or when σ changes (holding σ constant)? Differentiating $(5 - 3)$,

$$U_E = \frac{a+b}{E} - \frac{1}{3} \tag{5 − 4}$$

$$U_\sigma = \frac{a}{\sigma} - \frac{b}{(1-\sigma)} \tag{5 − 5}$$

(where I have used the fact that $\ln(AB) = \ln A + \ln B$). Note that U_E does not depend on σ, and U_σ does not depend on E. Also, U_E is a decreasing function of E,and U_σ is a decreasing function of σ. So each person has single–peaked preferences in each dimension : her preferences over total expenditure are single–peaked, for any given split σ between categories, and her preferences over the proportion σ going to police services are single–peaked, for any given total level E of expenditure.

Therefore the median voter theorem applies. If people vote on the total level of expenditure, the median of people's preferred levels of total expenditure will defeat any other expenditure level in a pairwise vote. If they vote over the division between categoties, then the median preferred level of σ will defeat any other proposed level of σ in a pairwise vote.

What are the preferred levels of the voters? A person with preference parameters a and b will want a level of E such that $U_E = 0$, and a level of σ such that $U_\sigma = 0$. From equations $(5 - 4)$ and $(5 - 5)$, then the preferred levels are

$$E = 3(a + b) \tag{5 − 6}$$

$$\sigma = \frac{a}{a+b} \tag{5 − 7}$$

From the data in question 4

$$a_1 + b_1 = 5 \quad ; \quad a_2 + b_2 = 6 \quad ; \quad a_3 + b_3 = 12$$

$$\frac{a_1}{a_1 + b_1} = 0.8 \quad ; \quad \frac{a_2}{a_2 + b_2} = 0.333 \quad ; \quad \frac{a_3}{a_3 + b_3} = 0.5$$

That means that the median preferred level of E is $3(6) = 18$,and the median preferred share σ is 0.5.

So if they vote separately on total expenditure E, and on the share σ of expenditure going to police services, then the result of the voting will be $Y = 9$, $Z = 9$.