AS/ECON 4080 Answers to Assignment 1 January 2006

$Q1$. What are all the efficient allocations in the following two–good, two–person economy?

Good X is a pure private good, and good Z is a pure public good. The economy's production possibility frontier has the equation :

$$X + Z^2 = 360$$

where X and Z are the total quantities produced of the private good and of the public good, respectively.

Person 1's preferences can be represented by the utility function

$$U^1(x_1, z_1) = x_1 + 50z_1 - 2(z_1)^2$$

and person 2's by the utility function

$$U^2(x_2, z_2) = x_2 + 30z_2 - (z_2)^2$$

where x_i is person i's consumption of the private good, and z_i is person i's consumption of the public good.

$A1$. Because the production possibility frontier has the equation $X + Z^2 = 360$, therefore $X = 360 - Z^2 \equiv F(Z)$, so that the marginal rate of transformation (MRT) is $-F'(Z)$, or

$$MRT = 2Z$$

For person 1, $MU^1_X = 1$ and $MU^1_Z = 50 - 4z_1$, so that

$$MRS^1 \equiv \frac{MU^1_Z}{MU^1_Z} = 50 - 4z_1$$

For person 2, $MU^2_X = 1$ and $MU^2_Z = 30 - 2z_2$ so that

$$MRS^2 \equiv \frac{MU^2_Z}{MU^2_X} = 30 - 2z_2$$

Since $z_1 = z_2 = Z$, since good Z is a pure public good, the Samuelson condition, $MRS^1 + MRS^2 = MRT$ here is

$$80 - 6Z = 2Z$$

or

$$Z = 10$$

Any allocation (x_1, x_2, Z) for which $x_1 \geq 0$, $x_2 \geq 0$, $Z = 10$, and $x_1 + x_2 = 300 - Z^2 = 260$ will be efficient.

$Q2$. What are all the efficient allocations in the following two–good, two–person economy?

Good X is a pure private good, and good Z is a pure public good. The economy's production possibility frontier has the equation :

$$X + Z = 60$$

Person 1's preferences can be represented by the utility function

$$U^1(x_1, z_1) = 2\ln x_1 + 8\ln z_1$$

and person 2's by the utility function

$$U^2(x_2, z_2) = \ln x_2 + \ln z_2$$

where x_i is person i's consumption of the private good, and z_i is person i's consumption of the public good.

$A2$. Because the production possibility frontier has the equation $X + Z = 60$, therefore $X = 60 - Z \equiv F(Z)$, so that the marginal rate of transformation (MRT) is $-F'(Z)$, or

$$MRT = 1$$

For person 1, $MU^1_X = \frac{2}{x_1}$ and $MU^1_Z = \frac{8}{z_1}$, so that

$$MRS^1 \equiv \frac{MU^1_Z}{MU^1_Z} = \frac{4x_1}{z_1}$$

For person 2, $MU^2_X = \frac{1}{x_2}$ and $MU^2_Z = \frac{1}{z_2}$ so that

$$MRS^2 \equiv \frac{MU^2_Z}{MU^2_X} = \frac{x_2}{z_2}$$

Since $z_1 = z_2 = Z$, since good Z is a pure public good, the Samuelson condition, $MRS^1 + MRS^2 = MRT$ here is

$$\frac{4x_1}{Z} + \frac{x_2}{Z} = 1$$

or

$$4x_1 + x_2 = Z \qquad (PAS)$$

Any allocation (x_1, x_2, Z), with all 3 consumption levels non–negative, which satisfies the feasibility condition

$$x_1 + x_2 + Z = 60 \qquad (feas)$$

and the Samuelson condition (PAS) will be efficient.

If we substitute for x_2 from equation $feas$ into equation PAS, it becomes

$$4x_1 + (60 - x_1 - Z) = Z$$

or

$$Z = 30 + \frac{3}{2}x_1 \qquad (2-3)$$

If $x_1 = 0$, then equation $(2-3)$ says that $Z = 30$, and the feasibility condition then says that $x_2 = 60 - 0 - 30 = 30$. So $(x_1, x_2, Z) = (0, 30, 30)$ is an efficient allocation. But if $x_1 = 10$, then equation $(2-3)$ says that $Z = 45$, and the feasibility condition $(feas)$ says that $x_2 = 60 - 10 - 45 = 5$. If x_1 increases to 12, then $Z = 48$ (from equation $(2-3)$) and $x_2 = 0$.

So any allocation (x_1, x_2, Z) will be efficient if (and only if) : (a) $0 \leq x_1 \leq 12$; (b) $Z = 30 + \frac{3}{2}x_1$; (c) $x_2 = 30 - \frac{5}{2}x_1$

*Q*3. What would the Lindahl equilibrium allocation be for the economy described in question #1 above, if the price of the private good were 1, if person #1's income were 240, and if person #2's income were 120?

*A*3. To find the Lindahl equilibrium, the demand functions of the two people for the public good must be calculated.

If person 1 had to pay p^1 per unit for the public good (and 1 per unit for the private good), then she would choose a consumption bundle (x_1, z_1) such that $MRS^1 = p^1$; that is, she would choose a consumption bundle so that her indifference curve was tangent to her budget line. From the answer to question 1, $MRS^1 = 50 - 4z_1$, which means that she would choose a consumption bundle such that

$$50 - 4z_1 = p^1 \qquad (3-1)$$

Equation $(3-1)$ defines the inverse demand function for the public good for person 1 : she would demand z_1 units of the public good at a price of p^1 only if $50 - 4z_1 = p^1$.

Similarly, person 2 would choose a consumption bundle such that

$$40 - 2z_2 = p^2 \qquad (3-2)$$

if she had to pay a price of p^2 per unit of the public good.

The answer to question #1 showed that $Z = 10$ in any efficient allocation. Therefore, we must have $z_1 = z_2 = Z = 10$ in the Lindahl equilibrium. Substituting into equations $(3-1)$ and $(3-2)$, the 2 people's Lindahl prices are $p^1 = 50 - 4(10) = 10$ and $p^2 = 30 - 2(10) = 10$.

Since $Z = 10$, each person's total tax bill would be 100. That would leave person 1 with $240 - 100 = 140$ to spend on the private good, and person 2 with $120 - 100 = 20$. Also notice that this resulting allocation — $x_1 = 140$, $x_2 = 20$, $Z = 10$ — is inside the production possibility frontier. Here $x_1 + x_2 + Z^2 = 260 < 360$. The reason for this is that the marginal cost of the public good (in terms of foregone private good production) slopes up : $-F'(Z) = 2Z$. Therefore, the revenue collected from Lindahl taxation exceeds the cost of the public good provision. The extra revenue could be given back to the 2 people to spend on the private good.

*Q*4. A city contains N identical people, each of whose preferences can be represented by the utility function

$$U(x, z) = x + a \ln z$$

where x is the person's consumption of a private good, z is the number of trips the person takes per day on a highway, and a will be defined below.

The parameter a measures the speed of traffic flow on the road, and is defined as

$$a = \sqrt{\frac{K}{Nz}}$$

where K is the size of the road. The total cost of a road of size K is cK.

If the city government could control how many trips each person takes per day on the highway z, and the size K of the road, what would be the efficient choices of z and K?

*A*4. If the city built a road of size K, and shared the cost among all the identical residents, then each resident would have $y - \frac{cK}{N}$ left to spend on the private good.

So the city government would like to maximize each resident's utility

$$y - \frac{CK}{N} + \sqrt{\frac{K}{Nz}} \ln z$$

with respect to K and z.

To do so, differentiate the above expression with respect to z, and with respect to K, and set the derivatives equal to 0, giving optimality conditions

$$\frac{\ln z}{2\sqrt{KNz}} - \frac{c}{N} = 0 \qquad (foc - K)$$

$$\sqrt{\frac{K}{N}}[\frac{1}{z}\frac{1}{\sqrt{z}} - \frac{1}{2}\frac{\ln z}{z^{3/2}}] = 0 \qquad (foc - z)$$

But equation $(foc - z)$ is equivalent to

$$\ln z = 2 \qquad (4 - 3)$$

or

$$z = e^2$$

where e is the base of natural logarithms.

Substituting from equation $(4 - 3)$ into equation $(foc - K)$ yields

$$K^{-1/2}N^{-1/2}\frac{1}{e} = \frac{c}{N} \qquad (4 - 4)$$

or

$$K = \frac{N}{(ce)^2} \qquad (4-5)$$

which is the efficient size of the road.

*Q*5. In the city described in question #4 above, suppose that the city government could not control directly the number of trips each person could take on the road, but could set a toll p for each trip taken by any person.

What toll should it charge? Would toll revenue cover the cost of the road?

*A*5. Suppose drivers had to pay a cost of p per trip on the road. The utility of a driver if she chose to take z trips would be

$$y - pz + a \ln z$$

where y is her income. She will treat a as a given constant, out of her control, if there are many people in the city : it is determined by the average number of trips everyone takes, and her own driving has very little influence on that.

So the number of trips z which maximizes her utility is the z which maximizes the above expression, that is the z which makes the derivative of the above expression equal 0 :

$$\frac{a}{z} = p \qquad (5-1)$$

Equation $(5-1)$ defines her demand for trips as a function of the toll she pays : $z = a/p$.

From question 4, if the road is optimal, then

$$z^* = e^2$$

$$K^* = \frac{N}{(ce)^2}$$

so that

$$a^* = \sqrt{\frac{K}{zN}} = \frac{1}{ce^2}$$

From equation $(5-1)$, the toll p which will make drivers choose z^* trips, when $a = a^*$ is

$$p^* = \frac{1}{ce^4}$$

How much revenue is collected, in total, from tolls? From equation $(5-1)$ it is $Np^*z^* = Na^*$, which equals

$$Np^*z^* = \frac{N}{e^2c}$$

The total cost of the road is

$$cK^* = \frac{N}{e^2c}$$

so the taxes collected exactly cover the cost of the road.