$Q1$. Suppose that a city is inhabited by many different people. The income of person i is y_i, which is assumed to be fixed, and unaffected by any taxes levied in the city.

The city can levy a proportional tax rate on the income of all its residents, at rate t.

The city's taxes are collected by a private agency, which receives a share of the revenue collected as compensation for collecting the taxes. This share is an increasing function of the tax rate t ; the agency collects a share s of all tax revenue collected,

$$s = \frac{\alpha}{2}t$$

where α is some positive constant.

The revenue collected from the income tax (after the collection agency has received its share) is distributed equally to all residents of the city.

Person i knows her own income, and knows the income of all the other people in the city.

What is her preferred tax rate t?

$A1$. The total revenue collected, if the proportional income tax rate is t, is tY, if Y is the total income of all residents of the city. But the collection agency takes a fraction s of the revenue. Since $s = \frac{\alpha}{2}t$, then the total amount of money taken by the collection agency, stY, equals

$$stY = \frac{\alpha}{2}t^2Y \qquad (1-1)$$

That means that the net income from the tax, available to distribute to residents, is $(1-s)tY$,

$$(1-s)tY = [t - \frac{\alpha}{2}t^2]Y \qquad (1-2)$$

What each person will get is $1/N$ times this net income, where N is the number of people. If

$$\bar{y} \equiv \frac{Y}{N}$$

is the average income in the city, then each person will get a share of the tax revenue (net of the collection agency's cut), of

$$[t - \frac{\alpha}{2}t^2]\bar{y} \qquad (1-3)$$

So what is the disposable income of a person, if she had to pay income tax at the rate t, but received a share of the tax revenue defined by expression $(1-3)$ above? This disposable income (call it x_i) is

$$x_i = (1-t)y_i + [t - \frac{\alpha}{2}t^2]\bar{y} \qquad (1-4)$$

or

$$x_i = y_i + t(\bar{y} - y_i) - \frac{\alpha}{2}t^2\bar{y} \qquad (1-5)$$

Person i's preferences over different tax rates t depend entirely on the level x_i of disposable income she gets. So to see how she ranks the different alternatives, differentiate expression $(1-5)$ with respeect to t :

$$\frac{\partial x_i}{\partial t} = [\bar{y} - y_i] - \alpha t \bar{y} \qquad (1-6)$$

Expression $(1-6)$ is positive if and only if

$$t < \frac{1}{\alpha}[1 - \frac{y_i}{\bar{y}}] \qquad (1-7)$$

So the person has single–peaked preferences : her disposable income increases with the tax rate t, as long as inequality $(1-7)$ holds, and then it decreases with the tax rate, as the tax rate increases above $(\bar{y} - y_i)/(\alpha \bar{y})$.

Her preferred tax rate is the tax rate is the one for which $\partial x_i / \partial t = 0$. If this preferred tax rate is denoted $t^*(y_i)$ then

$$t^*(y_i) = \frac{1}{\alpha}[1 - \frac{y_i}{\bar{y}}] \qquad (1-8)$$

Note that this preferred tax rate is positive if and only if her income is less than the average in the city : $t^*(y_i) > 0$ if and only if $y_i/\bar{y} < 1$ if and only if $y_i < \bar{y}$.

*Q*2. If the tax rate t for the city described in question #1 above is decided by direct vote of the residents, using pairwise majority rule, what tax rate will be selected?

*A*2. The median voter theorem applies here. The answer to question 1 showed that the preferences of each voter were single peaked, with a peak at $t^*(y_i)$, defined in equation $(1-8)$.

The median voter theorem then says that, under pairwise majority rule, the tax rate chosen will be the median of the voters' values for $t^*(y_i)$.

But note that $t^*(y_i)$ depends only on the person's income, and that it is a decreasing function of the person's income. This should not be surprising : each voter gets the same share of the tax revenue, but the low–income voters have to pay more taxes.

So the median preferred tax rate will be the preferred tax rate of the voter of median income : all voters of higher income will want lower tax rates, and all voters of lower income will want higher tax rates.

Therefore, the voters will choose a tax rate of

$$\frac{1}{\alpha}[1 - \frac{y^{med}}{\bar{y}}]$$

where y^{med} is the median income in the city.

Note that the tax rate chosen will be positive if and only if the median income y^{med} is less than the mean income $\bar{y}$. The shape of the income distribution in most cities (and countries) does seemed skewed this way, with a median income less than the mean.

$Q3$. Another city is identical to the one described in questions # 1 and 2, with two differences. Instead of distributing the net revenue from the income tax in cash, this city spends it all on a public good. Each person in the city has a utility function

$$u(x_i, Z) = x_i + \ln Z$$

where x_i is the person's income (net of tax), and Z is the level of the pure public good provided in the city.

Also, in this city the tax collection agency does not get a share of the tax revenue ; all the tax revenue collected is used for public good provision.

If the cost of the public good is c, what level of tax would be chosen in this city?

$A3$. If the income tax rate is t, then the total tax revenue collected will be tY, where Y is the aggregate income of all residents of the city. All this revenue will be spent on the public good. Since the cost per unit of the public good is c, therefore,

$$Z = \frac{tY}{c} \tag{3-1}$$

if the income tax rate is t.

If person i's income is y_i, then her net–of–tax income will be $(1-t)y_i$ if the proportional tax rate is t. So (from $(3-1)$), the utility of person i will be

$$U^i = (1-t)y_i + \ln\left(\frac{tY}{c}\right) \tag{3-2}$$

when the tax rate is t.

How does her utility vary with the tax rate t? Differentiate $(3-2)$ with respect to t to get

$$\frac{\partial U^i}{\partial t} = -y_i + \frac{1}{t} \tag{3-3}$$

(using the fact that the derivative of $\ln a$ with respect to a is $1/a$).

Equation $(3-3)$ shows that the person's utility increases with t as long as $1/t < y_i$, or as long as $t < t^*(y_i)$, where

$$t^*(y_i) = \frac{1}{y_i} \tag{3-4}$$

Again, preferences are single–peaked : each person's utility goes up with the tax rate t as long as the tax rate is less than her preferred rate $t^*(y_i)$, and then her utility decreases as t increases above $t^*(y_i)$.

As well, a voter's preferred tax rate $t^*(y_i)$ here is a **decreasing** function of her income y_i. [That is because the preferences defined in this question imply that the income elasticity of demand for the public good is 0. Higher income people have to pay a higher share of the cost of increases in the public good, and their willingness to pay for the public good is no higher than low–income people's.]

So the tax rate which will be chosen under pairwise majority rule is the preferred tax rate of the voter of median income : voters of higher income want lower tax rates and voters of lower income want higher tax rates. The tax rate chosen would be

$$t^*(y^{med}) = \frac{1}{y^{med}} \tag{3 – 5}$$

where y^{med} is the median income of the city's residents.

*Q*4. Three towns are located along a straight road. Town 1 is at the south end of the road, and has 10000 inhabitants. Town 2 is 1ocated 9 kilometres north of town 1, and has 5000 inhabitants. Town 3 is located 21 kilometres north of town 2, and has 12000 inhabitants.

People can travel along the road at 1 kilometre per minute.

The three towns are all in the same county (and contain all the people in the county). Two parties are running for election to the county council ; each wants to win. The one issue in the election is where to locate the new events centre for the county. Each voter wants her travel time to the events centre to be as low as possible.

What location will each party propose as a location for the event centre, if they each want to win the election?

What location would minimize the aggregate travel time of all 27000 residents of the county?

*A*4. Each voter's preferred location for the events centre is in the town in which she lives : that minimizes her own travel costs. Her preferences over location are also single–peaked : the further away the events centre is from her town, the higher her travel costs.

Since there are two parties contesting the election, and since each wants to win the election, Hotelling's principle of minimum differentiation applies here. In equilibrium, both parties will support the median of the policies. Here, that means each party will propose builidng the events centre at the median of the preferred locations. Here, that median location is in town 2. Any other location would lose an election to that choice. If someone proposed a location north of town 2, then it would be defeated by a coalition of the 10000 people in town 1, and the 5000 people in town 2. If someone proposed a location south of town 2, then the location in town 2 would be supported by all voters in towns 2 and 3, and would win the election.

So the principle of minimum differentiation implies that town 2 will be the location supported by both parties.

What is the aggregate travel time of all 27000 residents of the county? People can travel at 1 kilometre per minute.

if the events centre is located in town 2, then the aggregate travel time of all residents is $9(10000) + 21(12000)$ minutes, or 342000 minutes. That's because people in town 1 would have to travel 9 kilometres, and people in town 3 21 kilometres, and people in town 2 don't have to travel.

Suppose we proposed moving the events centre a kilometre north of town 2. Then travel costs of residents of town 3 would go down, by a minute each. Travel costs of residents of town 1 would increase, by a minute each. But now residents of town 2 would also have to travel for a minute. The aggregate increase in travel time would be $10000 + 5000 - 12000 = 3000$ minutes. Moving the events centre a mile north of town 2 would increase aggregate travel time, added up over all residents. Similarly moving it 2 miles north would increase aggregate travel costs by 6000 minutes, and so on.

What if we moved the events centre a kilometre south of town 2? Now travel costs of residents of town 1 would decrease (by a minute per person), travel costs of residents of town 3 would increase (by 1 minute each), and residents of town 2 would now have to travel (for a minute). The total change in aggregate travel time would be $12000 + 5000 - 10000 = 7000$ minutes. Similarly, moving the events centre 2 kilometres south of town 2 would increase aggregate travel time by 14000 minutes, and so on.

Therefore, moving the events centre in either direction away from town 2 would increase aggregate travel costs. The best location, if we want to minimize the total travel costs added up over all residents of the county, is right in town 2.

*Q*5. Redo question # 4 if residents' travel times were proportional to the **square** of the distance travelled?

*A*5. Nothing is changed in the political equilibrium. Even though travel costs are now proportional to the square of the distance travelled, rather than to the distance itself, each voter has single–peaked preferences over location, and has a preferred location in her own town. The Hotelling principle of minimum differentiation applies, and the winning policy for a political party (if there are only 2 political parties) is to propose a location in town 2 for the events centre.

Now what location minimizes the aggregate travel costs? Let x be the location of the events centre, measured from the south end of the road. Then travel times will be proportional to x^2 for residents of town 1, $(x-9)^2$ for residents of town 2, and $(x-30)^2$ for residents of town 3.

Aggregate travel costs are therefore proportional to

$$A(x) = 10000x^2 + 5000(x-9)^2 + 12000(x-30)^2 \qquad (5-1)$$

Taking the derivative with respect to location x,

$$A'(x) = 1000[20x + 10(x-9) + 24(x-30)] = 2000[27x - 405] \qquad (5-2)$$

That means that $A'(x) = 0$ when $27x = 405$, or

$$x = 15$$

This location is indeed a minimum, not a maximum, for total travel costs, since

$$A''(x) = 54000 > 0$$

The optimal location is therefore 15 kilometres north of town 1, putting it 6 kilometres north of town 2 (and 15 kilometres south of town 3).

Notice that changing the form of the travel cost function does affect the "efficient" location choice (if "efficient" means "total cost minimizing"), but it does not affect the location choice of vote–maximizing parties. In this sense, the equilibrium outcome may not be efficient, since political parties do not take into account the fact that moving the events centre further south from the centre imposes greater costs an each person living in town 3 (very far away) than it imposes on each person in town 2 (fairly close).