*Q*1. What are all the efficient allocations in the following two–good, two–person economy?

Good X is a pure private good, and good Z is a pure public good. The economy's production possibility frontier has the equation :

$$X + 2Z = 15$$

where X and Z are the total quantities produced of the private good and of the public good, respectively.

Person 1's preferences can be represented by the utility function

$$U^1(x_1, z_1) = x_1 + 3 \ln z_1$$

and person 2's by the utility function

$$U^2(x_2, z_2) = x_2 z_2$$

where x_i is person i's consumption of the private good, and z_i is person i's consumption of the public good, and where "$\ln z_1$" is the natural logarithm of z_1.

*A*1. When there is one pure private good, and one pure public good, an allocation will be efficient if — and only if — it has the following properties :

(i) the production plan (X, Z) is on the production possibility frontier

(ii) the sum of people's private good consumption equals the total quantity X produced of the private good

(iii) each person consumes the total quantity Z available of the public good

(iv) the sum of the people's marginal rate of substitution between the public good and the private good equals the marginal rate of transformation

(v) each person's private good consumption, and the each person's public good consumption are non–negative

Property (iv), the Samuelson condition, is the main piece of calculation in this question. To see if an allocation satisfies this condition, the marginal rates of substitution, and the marginal rate of transformation have to be derived.

The marginal rate of transformation (MRT) here is 2, since the equation of the production possibility frontier is $X + 2Z = 15$: increasing Z by one unit implies that X must fall by 2 units, so that the opportunity cost of one more unit of the pure public good is 2 fewer units of the pure private good. [Alternatively, the equation of the production possibility frontier can be written $X = 15 - 2Z$, so that $dX/dZ = -2$.]

A person's marginal rate of substitution is the ratio of the marginal utilities of the 2 goods. It is important here that the ratio is not upside down. The MRS to be used here is the marginal utility of the pure public good, divided by the marginal utility of the pure private good. [Why? It's how much a person would be willing to pay, in units of the private good, for a little more of the public good.]

For person 1, the following mathematical fact has to be used : the derivative of the natural logarithm of z_1 with respect to z_1 is $1/z_1$. So for person 1,

$$MU_x^1 = 1 \quad MU_z^1 = \frac{3}{z_1} \qquad MRS_1 = \frac{MU_z^1}{MU_x^1} = \frac{3}{z_1}$$

where MU_x^1 and MU_z^1 are her marginal utilities for the private good and the public good respectively.

For person 2,

$$MU_x^2 = z_2 \quad MU_z^2 = x_2 \qquad MRS_2 = \frac{MU_z^2}{MU_x^2} = \frac{x_2}{z_2}$$

Since everyone should consume all the public good available [since it's a good and not a bad] (this is property (iii) above), $z_1 = z_2 = Z$, so that the Samuelson condition $MRS_1 + MRS_2 = MRT$ becomes

$$\frac{3}{Z} + \frac{x_2}{Z} = 2$$

or

$$3 + x_2 = 2Z \tag{1-1}$$

Properties (i), (ii) and (iii) above imply that any allocation (x_1, x_2, Z) must have $(x_1 + x_2, Z)$ on the production possibility frontier, or

$$x_1 + x_2 + 2Z = 15 \tag{1-2}$$

And that pretty well describes the efficient allocations ; any allocation (x_1, x_2, Z) satisfying $(1-1)$ and $(1-2)$ will be efficient [if $x_1 \geq 0$ and $x_2 \geq 0$ and $Z \geq 0$].

These conditions can be simplified somewhat by substituting for Z from equation $(1-2)$ into equation $(1-1)$ to get

$$x_2 = 6 - \frac{x_1}{2} \tag{1-3}$$

$$Z = \frac{9}{2} - \frac{x_1}{4} \tag{1-4}$$

So the following gives a complete description of all the efficient allocations :

— pick any x_1 between 0 and 12 ; then pick x_2 and Z according to $(1-3)$ and $(1-4)$

[Why 12 as the maximum value for x_1? Because then $x_2 = 0$]

The following table illustrates some of the efficient allocations : note that they all satisfy the 5 properties for efficiency described above.

x_1	x_2	$z_1 = z_2 = Z$
0	6	4.5
2	5	4
6	3	3
9	1.5	2.25
12	0	1.5

*Q*2. What are all the efficient allocations in the following two–good, two–person economy?

Good X is a pure private good, and good Z is a pure public good. The economy's production possibility frontier has the equation :

$$X + 2Z = 15$$

where X and Z are the total quantities produced of the private good and of the public good, respectively.

Person 1's preferences can be represented by the utility function

$$U^1(x_1, z_1) = x_1 + 3\ln z_1$$

and person 2's by the utility function

$$u^2(x_2, z_2) = \ln x_2 + \ln z_2$$

where x_i is person i's consumption of the private good, and z_i is person i's consumption of the public good.

*A*2. This is a bit of a trick question. The only item that is changed here from question #1 is the utility function for person 2. But if the MRS for person 2 is computed,

$$MU_x^2 = \frac{1}{x_2} \quad MU_z^2 = \frac{1}{z_2} \qquad MRS_2 = \frac{MU_z^2}{MU_x^2} = \frac{x_2}{z_2}$$

which is the same expression for person 2's MRS as was derived in question #1.

What has happened? The utility function u^2 is a *monotonic transformation* of the utility function U^2 of question #1 : u^2 is the natural logarithm of U^2 (since $\ln(xz) = \ln x + \ln z$). The indifference curves for the utility function u^2 look exactly the same as the indifference curves for the utility function U^2. For efficiency (with or without a pure public good), it is the shape of the indifference curves which matters, not the value of the utility. [See chapter 4 of Varian for a discussion of why the *cardinal* value of utility is not important here, just the preferences the utility functions represent.]

So the answer is that the efficient allocations for question #2 are exactly the efficient allocations for question #1.

$Q3$. What would the Lindahl equilibrium be in the economy described in question #1 above, if person #1 had an income (measured in units of the private good) of 9, and person #2 had an income of 6?

$A3$. At the Lindahl equilibrium,

$$p_1(Z) + p_2(Z) = MRT$$

where $p_i(Z)$ is the price person i is willing to pay for a little more of the public good [in units of the private good], as a function of the quantity she is consuming of the public good (that is, $p_i(Z)$ is the height of person i's demand curve for the public good, corresponding to a quantity of Z).

So to answer the question, the people's demand curves for the public good need to be derived.

From econ 2300, the person's demand curve results from her finding the highest possible indifference curve on her budget line, with optimality condition that the person's MRS between any two goods equal the price ratio. So if the price of the private good is taken as numéraire, then a person's MRS equals the price ratio if

$$MRS_i = p_i$$

where p_i is the person's Lindahl price.

For person 1, the demand curve is easy to calculate. From the answer to question #1, her MRS equals $1/Z$, so that her demand function for the public good is defined by the condition $MRS_1 = 1/Z = p_1$ which means that her "personalized" Lindhal price is defined by

$$p_1(Z) = \frac{1}{Z} \qquad (3-1)$$

[Note that this demand curve does not shift with changes in her income, since her preferences are *quasi–linear* (as in section 6.3 of Varian's 2300/2350 text).]

It's a little more complicated for person 2. From the answer to question #1,

$$MRS_2 = \frac{x_2}{Z} = p_2 \qquad (3-2)$$

but equation $(3-2)$ is not a demand function, since it depends on the person's consumption x_2 of good X, which is something he is choosing. However, if person 2 had income of y_2, then her demands must be on her budget line,

$$p_x x_2 + p_2 Z = y_2$$

where p_x is the price of the private good. Since the private good is being used as a numéraire, $p_x = 1$, so that his budget constraint can be written

$$x_2 + p_1 Z = y_2 \qquad (3-3)$$

Together, equations $(3-2)$ and $(3-3)$ define the person's demand functions. $(3-3)$ implies that $x_2 = y_2 - p_2 Z$; substituting this in $(3-2)$ yields

$$\frac{y_2 - p_2 Z}{Z} = p_2$$

or

$$Z = \frac{1}{2}\frac{y_2}{p_2} \tag{3-4}$$

which is the person's demand function for the public good. [This should be familiar from econ 2300 ; see the appendix to chapter 5 in Varian's text, for example.]

In this question, person 2's income is 6, so that $(3-4)$ can be written

$$p_2 = \frac{3}{Z} \tag{3-5}$$

Here the MRT equals 2 (from the answer to question #1), so that, at the Lindahl equilibrium

$$p_1(Z) + p_2(Z) = 2$$

Using $(3-1)$ and $(3-5)$, at the Lindahl equilibrium

$$\frac{3}{Z} + \frac{3}{Z} = 2 \tag{3-6}$$

or

$$Z = 3$$

So, at the Lindhal equilibrium, each person faces a Lindahl price of 1. Each person consumes 3 units of the public good, and pays taxes of $t_i = p_i Z = 3$, leaving private consumption of $x_1 = y_1 - t_1 = 9 - 3 = 6$ and $x_2 = y_2 - t_2 = 6 - 3 = 3$, so that at the Lindahl equilibrium $x_1 = 6, x_2 = 3, Z = 3$. [The answer to question #1 shows that this Lindahl equilibrium is one of the (many) efficient allocations.]

*Q*4. What would the (Nash equilibrium) outcome be in the economy described in question #1 above, if person #1 had an income (measured in units of the private good) of 11, and person #2 had an income of 4, and if the public good were provided by voluntary donations from the two people, if the two people acted non–cooperatively?

*A*4. If the two people act selfishly, and if they do not cooperate, then each person wants to donate up to the point where her MRS equals the MRT. For person 1,

$$MRS_1 = \frac{3}{Z} \tag{4-1}$$

If person 1 is willing to donate at all, she will donate just enough so that the total quantity of the public good Z implies that $MRS_1 = 3/Z = MRT = 2$ or

$$Z = \frac{3}{2} \tag{4-2}$$

For person 2,

$$MRS_2 = \frac{x_2}{Z} \tag{4-3}$$

He will want to donate just up to the point at which $MRS_2 = MRT = 2$. Since equation $(4-2)$ already defines Z, $MRS_2 = 2$ if and only if $x_2 = 3$.

So, at an equilibrium in which both people contribute, $Z = 3/2$ and $x_2 = 3$. Since person 2's income is 4, $x_2 = 3$ implies that she donates \$1 towards provision of the public good. The total cost of the public good is $2Z = 2(\frac{3}{2}) = 3$, so that person 1 donates \$2, and has $11 - 2 = 9$ left for spending on the public good.

Therefore, at the Nash equilibrium, person 1 donates 2, person donates 1, and the quantity provided of the public good is $(1 + 2)/2 = 1.5$. This outcome is not efficient : at this equilibrium $MRS_1 + MRS_2 = 2 + 2 = 4 > 2$, so that "too little" of the public good is provided.

*Q*5. What would the (Nash equilibrium) outcome be in the economy described in question #1 above, if person #1 had an income (measured in units of the private good) of 7, and person #2 had an income of 8, and if the public good were provided by voluntary donations from the two people, if the two people acted non–cooperatively?

*A*5. The "income transfers don't matter" theorem says that any redistribution of income results in exactly offsetting changes in donations. But this theorem holds only if all people are actually contributing positive amounts to the provision of the public good.

In this question, person 1's income has gone down by 4, and person 2's has gone up by 4 [from the income levels in question #4], so that — if the theorem applied — person 1's equilibrium contributions would go down by 4, and person 2's would go up by 4. But then [from the answer to question #4] person 1 would be donating $2 - 4 = -2$, which is a negative amount.

So when $y_1 = 7$ and $y_2 = 8$, one of the people does not make positive donations in equilibrium. Person 1 donates nothing. Person 2 donates up to the point at which $MRS_2 = 2$. Since $MRS_2 = x_2/Z$, this means

$$\frac{x_2}{Z} = 2 \tag{5-1}$$

Since person 2 has an income of 8, $x_2 = 8 - 2Z$ if all the funding for the public good comes from her donations. (Each unit of the public good reduces her income by 2.) Equation $(5-1)$ then becomes

$$\frac{8 - 2Z}{Z} = 2 \tag{5-2}$$

or

$$Z = 2 \tag{5-3}$$

Is this a Nash equilibrium, in which person 2 donates \$4 and person 1 donates nothing, so that $Z = 4/2 = 2$? At this outcome $x_2 = 8 - 4 = 4$ and $x_1 = 7$.

Here

$$MRS_1 = \frac{1}{Z} = \frac{1}{2} \tag{5-4}$$

Person 1's MRS is less than the MRT (which is 2), so that she does not want to donate anything.

Therefore, the Nash equilibrium in this question has only person #2 contributing, and person #1 free–riding on his contribution. At this equilibrium $MRS_1 + MRS_2 = 2.5 > 2 = MRT$ so that, again, the equilibrium is inefficient, due to an under–provision of the public good.