The following information is to be used in all the questions.

Farm #1 is a dairy farm. It sells the milk it produces in a perfectly competitive market at a price of \$240 per truckload of milk that it produces. The cost to farm #1 of producing M truckloads of milk is

$$C^1(M) = M^2$$

(so that farm #1's profits are $240M - M^2$ if it produces M truckloads of milk).

Farm #2 is a wheat farm, which sells its wheat in a perfectly competitive market for \$360 a bushel. Farm 2's costs of producing W bushels of wheat are

$$C^2(W; M) = 2W^2 + 2WM$$

(so that farm #2's profits are $360W - 2W^2 - 2WM$). Farm #2's costs increase with farm #1's milk production, since more milk production requires more cows, and the cows tend to damage the wheat.

*Q*1. What are the efficient quantities of milk production and wheat production for the two farms?

*A*1. Since both farms are perfect competitors, the efficient outcome is the outcome which maximizes the total profits of the two farms together. From the data given above, the profits of the two farms can be written

$$\pi_1 = 240M - M^2 \qquad (1-1)$$

$$\pi_2 = 360W - 2W^2 - 2WM \qquad (1-2)$$

so that

$$\pi_1 + \pi_2 = 240M + 360W - M^2 - 2W^2 - 2WM \qquad (1-3)$$

The efficient quantities of M and W are the quantities which maximize expression $(1-3)$. So the quantities of milk and wheat which maximize joint profits are those for which $\frac{\partial(\pi_1+\pi_2)}{\partial M} = 0$ and $\frac{\partial(\pi_1+\pi_2)}{\partial W} = 0$, so that (from equation (1 − 3))

$$\frac{\partial(\pi_1 + \pi_2)}{\partial M} = 240 - 2M - 2W = 0 \tag{1-4}$$

$$\frac{\partial(\pi_1 + \pi_2)}{\partial W} = 360 - 4W - 2M = 0 \tag{1-5}$$

Subtracting equation (1 − 4) from equation (1 − 5),

$$120 - 2W = 0 \tag{1-6}$$

so that the efficient quantity of wheat production is

$$W^* = 60 \tag{1-7}$$

Substituting for W^* in equation (1 − 4)

$$240 - 2M - 120 = 0 \tag{1-8}$$

so that the efficient quantity of milk production is

$$M^* = 60 \tag{1-9}$$

*Q*2. If farm #1 chose its milk output M to maximize its own profit, and did not negotiate with farm #2, what quantity M would farm #1 choose, and what quantity W of wheat would farm #2 choose to to produce?

*A*2. If farm #1 totally ignored farm #2, then it would choose its level of milk production M so as to maximize its own profit π_1, so that it would find the level of M for which

$$\frac{\partial \pi_1}{\partial M} = 240 - 2M = 0 \tag{2-1}$$

or a level of milk production M^{eq} for which

$$M^{eq} = 120 \tag{2-2}$$

Farm 2, if it cannot negotiate with farm 1, has to choose its own wheat production level to maximize its own profits, taking firm 1's milk production M^{eq} as given. So firm 2 picks W to maximize π_2, so that

$$\frac{\partial \pi_2}{\partial W} = 360 - 4W - 2M^{eq} = 360 - 4W - 240 = 0 \tag{2-3}$$

implying that its wheat production in equilibrium (when farm #1 has the property right, and when the firms do not negotiate) is

$$W^{eq} = 30 \qquad (2-4)$$

*Q*3. If farm #1 had to compensate farm #2 for any damage done by its cows, and if the two farms could not negotiate with each other, what quantity M would farm #1 choose, and what quantity W would farm #2 choose?

*A*3. How much compensation would farm #1 have to pay to farm #2 to compensate fully for the damage done by its cows?

If there were no cows to damage the wheat, farm #2's net profits would be

$$\pi_2^0 = 360 - 2W^2 \qquad (3-1)$$

since this is its profit when $M = 0$. The dollar amount of damage done by the cows is the difference between farm #2's actual profit π_2 (defined by equation $(1-2)$ above) and the profit π_2^0 it would earn if there were no cows. So the compensation farm #1 would have to pay to farm #2, in order to compensate exactly for the damage done by the cows is

$$C = 2WM \qquad (3-2)$$

which is the difference between π_2^0 and π_2. Note that C depends on both the milk production of farm #1, and on the wheat production of farm #2.

If farm #1 is legally required to pay compensation to farm #2, then its net earnings (net of payments it must make to farm #2) is

$$\pi_1 - C = 240M - M^2 - 2WM \qquad (3-3)$$

Farm #2's net earnings are its profits, plus the compensation it receives from farm #1,

$$\pi_2 + C = 360W - 2W^2 - 2WM + C = 360W - 2W^2 = \pi_2^0 \qquad (3-4)$$

If the two farms cannot negotiate with each other, farm #1 will choose its level of milk production to maximize its net earnings, so that it picks a level of milk production M^0 such that

$$\frac{\partial(\pi_1 - C)}{\partial M} = 240 - 2M - 2W = 0 \qquad (3-5)$$

and farm #2 will choose a level of wheat production which maximizes its own net earnings, so that it chooses a level W^0 of wheat production such that

$$\frac{\partial(\pi_2 + C)}{\partial W} = 360 - 4W = 0 \tag{3-6}$$

Note that farm #1's preferred level of milk production depends on how much wheat farm #2 is producing (since W affects the compensation it must pay), but that farm #2's preferred level of wheat production does not depend on M (since the compensation means that farm #2 can act as if there were no damage being done by the cows). From equation $(3-6)$, farm #2 chooses a wheat production level such that'

$$W^0 = 90 \tag{3-7}$$

Substituting $W = 90$ into $(3-5)$, farm #1 chooses a milk production level such that

$$240 - 2M - 180 = W \tag{3-8}$$

or

$$M^0 = 30 \tag{3-9}$$

*Q*4. Suppose that farm #1 had to pay a tax of $120 per truckload of milk to the government. This tax revenue does not go to the owners of farm #2, but to the general government revenue (and will not be spent on any government projects which give any benefit to farm #1 or farm #2).

But, unlike the situation in questions #2 or #3, the two farms are now capable of negotiating with each other. [Negotiation here does not alter the tax policy : farm #1 must still pay a tax of $120 for every truckload of milk that it produces.]

What quantities M and W will they agree to produce, after negotiating with each other?

*A*4. In this situation, farm #1 must pay the government a tax of

$$T = 120M \tag{4-1}$$

so that its net earnings are $\pi_1 - T$. But now farm #2 is not getting any of this tax revenue ; its net earnings are just π_2.

Since the tax revenue collected from farm #1 is not being paid back to either farm, the total net earnings of the two farms together are $\pi_1 + \pi_2 - T$, or

$$\pi_1 + \pi_2 - T = 240M - M^2 + 360W - 2W^2 - 2WM - 120M = 120M - M^2 + 360W - 2W^2 - 2WM \qquad (4-2)$$

If the two farms can negotiate, they will reach an agreement to choose milk and wheat production levels which maximize their total net earnings $\pi_1 + \pi_2 - T$. If farm #1 agrees to reduce its milk production, it will see its tax liabilities fall (in addition to getting a bribe from farm #2). So negotiation should lead to the farms agreeing to levels of M and W which maximize $\pi_1 + \pi_2 - T$.

How do changes in M and W affect this total? Differentiating expression $(4-2)$,

$$\frac{\partial(\pi_1 + \pi_2 - T)}{\partial M} = 120 - 2M - 2W \qquad (4-3)$$

$$\frac{\partial(\pi_2 + \pi_2 - T)}{\partial W} = 360 - 4W - 2M \qquad (4-4)$$

Now if both derivatives, $(4-3)$ and $(4-4)$ equalled zero, then we would have [subtracting expression $(4-3)$ from expression $(4-4)$ and setting it equal to 0], $W = 120$, which would imply (substituting $W = 120$ back into $(4-3)$ and setting that derivative equal to 0) that $M = -60$: a negative level of milk production!

Milk production cannot be negative. What is happening here is that the taxation of milk production now means that total earnings of the 2 farms together are maximized by having farm #1 cease production altogether. The optimum, from the viewpoint of the two farms, is to set $M = 0$, and then to choose W so as to maximize π_2. Substituting $M = 0$ into $(4-4)$, and setting the expression equal to 0, implies

$$360 - 4W = 0 \qquad (4-5)$$

or

$$W = 90$$

.

Note that when $W = 90$ and $M = 0$, then expression $(4-3)$ equals $120 - 180 = -60 < 0$. If $W = 90$, any increase in M above 0 will actually reduce the combined earnings of the 2 farms.

[How would the 2 farms negotiate this outcome? The net earnings of farm #1 are $240M - M^2 - 120M = 120M - M^2$. So, if farm #1 simply ignored farm #2, it would choose an output level of $M = 60$ (which maximizes $120M - M^2$ with respect to M). It would have net earnings of $120(60) - (60)^2 = 3600$.

Now how much would farm #2 gain, if farm #1 were to shut down? The damage done by the cows to farm #2 is $2WM$, which equals $2(90)(60) = 10800$ when $W = 90$ and $M = 60$. So if farm #1 were to shut down, farm #2's profit would go up by \$10800, which is (much) greater than farm #1's net earnings of \$3600. Farm #2 could offer a bribe to farm #1 to go out of business. If that bribe is between \$3600 and \$10800, then farm #1 is better off taking the bribe and going out of business, and farm #2 is better off offering the bribe and getting rid of the cows.

You can check, in fact, that, when $W = 90$, that $2WM > 120M - M^2$ for **any** $M > 0$; no matter what farm #1 does, the gain to farm #2 of having farm #1 shut down is bigger than the lost earnings of farm #1.]

*Q*5 Rank the outcomes in questions 1 through 4, in order of their efficiency.

*A*5 The measure of efficiency here is the sum of the profits of the two firms. [In the situation of question #4. the taxes paid to the government should **not** be subtracted off the profits, in calculating the overall measure of efficiency ; the farms' owners may not get the money, but someone does. However, this issue turns out not to matter here, since no taxes actually get collected, since the farms negotiate a deal to shut down farm #1.]

The table below indicates the outcomes in the 4 situations. Note that π_1 and π_2 are not necessarily what the owners of the farms get ; farm #2 will have to pay a bribe to farm #2 in the outcome for question #4, for example. But $\pi_1 + \pi_2$ is the measure of how efficient the outcome is, in each case : the value of the output produced, minus the cost of producing it.

question#	M	W	π_1	π_2	$\pi_1 + \pi_2$
1	60	60	10800	7200	18000
2	120	30	14400	1800	16200
3	30	90	6300	10800	17100
4	0	90	0	16200	16200

So, as it must be, the efficient outcome (question #1) ranks highest. The situation in question #3 is better than the situation in question #2, which is tied for worst with the situation inn question #4.