Preference Revelation : (b) A Project of Variable Size

In the example in the previous section, the project being considered had a fixed size, so that the public good provision decision was an "all or nothing" decision : either build the facility or don't build the facility.

In this section, a somewhat more complicated problem is considered : determining the quantity to provide of a pure public good. So, unlike the project considered in the previous section, the problem considered here is how much to provide of a pure public good, that is, how to implement the Samuelson rule when people's preferences are not known (except by the people themselves).

The variable being chosen here is the quantity Z of some pure public good. It is assumed that each person has a downward–sloping demand curve for the public good, a demand curve which the person knows, but which no–one else knows. So now the preference revelation mechanism must get people to announce their willingness to pay for the public good, as a **function** of the quantity Z provided of the public good.

Each person i will be asked to report her willingness to pay $v_i(Z)$ for the public good. $v_i(Z)$ is the amount, in dollars, that the person would be willing to pay for a little more of the public good, if a quantity Z is provided. Or it's what person i says is what she is willing to pay : we have no way of verifying whether what she reports is her true willingness to pay schedule, or not. As in the pivot tax of the previous section, the preference revelation mechanism here uses a fairly complicated tax rule, which should induce each person to report truthfully her willingness–to–pay schedule, if she understands the tax rules, and if she wants to manipulate the system to her own advantage.

The notation used here will be quite similar to the notation used in the previous sub–section : the main difference is that now people have to report a **whole demand curve**, not just a single number. So $v_i(Z)$ will denote the demand schedule which person i reports, some downward–sloping function, representing what she says she would be willing to pay for a little more of the public good, as a function of the amount provided. This $v_i(Z)$ denotes person i's **reported** MRS function. As in the previous section, we cannot tell whether person i is telling the truth or not — but we'd like to make it worth her while to tell the truth.

The marginal cost of each unit of the public good will be denoted c. That's just the MRT.

The government wants to provide an efficient allocation. So the quantity Z^* which it will actually provide will depend on the demand schedules reported by the people, and it will obey the Samuelson rule, for the **reported** demand schedules. That is, once all the people have reported their demand schedules for the public good, then the level that will actually get provided is the solution Z^* to the equation

$$v_1(Z^*) + v_2(Z^*) + \cdots + v_N(Z^*) = c \tag{1}$$

when the N people report their demand schedules. Equation (1) is the Samuelson condition, except with people's **reported** MRS's, $v_i(Z)$ used, since we don't actually know their true MRS

schedules.

As in the simple "pivot tax" mechanism with a project of fixed size, a person's tax liabilities here will have two parts. First of all, part 1 of the tax is simply the person's share of the cost of the public good. So each of the N people has to pay a fraction $1/N$ of the cost of the public good. If T denotes this first part of the tax, then for each person i,

$$T = \frac{cZ^*}{N} \tag{2}$$

since the public good costs c per unit, Z^* units are being provided, and the cost is being divided among all of the N people.

The second part of each person's tax is again a "pivot tax", determined by how the person's reported willingness–to–pay schedule alters the provision of the public good.

As in the previous section, consider person 1's taxes, and her incentives to manipulate the system. As before, let $\tilde{V}_1$ denote the sum of **everyone else's** announced willingness to pay.

$$\tilde{V}_1(Z) \equiv v_2(Z) + v_3(Z) + \cdots + v_N(Z)$$

So the Samuelson rule (1), which the government has promised to use, can now be written

$$v_1(Z^*) + \tilde{V}_1(Z^*) = c \tag{3}$$

As before, consider what level of the public good would be provided, if person 1's announced preferences were not taken into account, but her contribution to the cost were also left out. So define the level of public good provision $\tilde{Z}_1$ by the equation

$$\tilde{V}_1(\tilde{Z}_1) \equiv v_2(\tilde{Z}_1) + v_3(\tilde{Z}_1) + \cdots + v_N(\tilde{Z}_1) = \frac{N-1}{N}c \tag{4}$$

In figure 1, the horizontal dotted line represents the MRT. But the lower horizontal green dashed lines the MRT minus person 1's contribution. The downward–sloping green dashed line is the vertical sum of everyone else's announced demand curves. In the diagram, $\tilde{Z}_1$ equals 5, where the vertical sum of everyone else's announced demand curves crosses a line with a height $(N-1)/N$ times the MRT.

Notice that person 1 does not get to affect $\tilde{Z}_1$: $\tilde{Z}_1$ is determined only by the announced demands of **other people**.

The second part of person 1's tax, the pivot tax, is the area between the vertical sum of everyone else's announced demand curves, and their share $[N-1)/N]c$ of the costs, between the level $\tilde{Z}_1$ of the public good which would be provided without person 1, and the level Z^* which is actually provided. In figure 1, it's the triangle outlined in green, and labelled PT (for "pivot tax").

Mathematically, the area under a curve is measured using the integral of the function. So

$$PT = \int_{\tilde{Z}_1}^{Z^*} [\frac{N-1}{N}c - \tilde{V}_1(Z)]dZ \tag{5}$$

In figure 1, Z^* is to the right of $\tilde{Z}_1$: $\tilde{Z}_1 = 5$ and $Z^* = 6$ in that figure. That's because, in this example, person 1 seems to have a relatively strong demand for the public good, so including her valuation pushes up the quantity chosen. But that need not be the case : figure 2 illustrates a case in which person 1 announces a lower valuation than the average of the other people, so that including her actually reduces the quantity provided. In that case, there is still a pivot tax PT in the figure : again its the area between $\tilde{Z}_1$ and Z^*, between the vertical sum of everyone else's announced willingness to pay, and their shares of the cost. So formula (5) applies whether or not Z^* is greater than $\tilde{Z}_1$.

Person 1's total taxes are just the sum of the first part of her taxes, which is her share of the cost, and the second part, which is her pivot tax. That is, in figure 1, her taxes are the triangle labelled PT, plus the red rectangle labelled "tax of person 1 : part 1".

This tax PT really is a generalization of the pivot tax used in the previous section. When including person 1 affects the level of public good provided, she is assessed this extra tax. Except with a variable public good, she'll always (except by extreme coincidence) have some affect on the quantity provided, so that she'll aways have some pivot tax to pay.

How is person 1's pivot tax affected by her reported demand schedule? To answer that, consider how PT is affected by the quantity Z^* of the public good which is actually provided. That is, how does the value of PT in expression (5) change with the level Z^* of public good provision?

To answer that, remember the fundamental theorem of calculus, that the derivative of the integral of a function is just the function itself :

$$\frac{\partial}{\partial x}\int_{x_0}^{x} f(t)dt = f(x)$$

Here, equation (5) then says that

$$\frac{\partial PT}{\partial Z^*} = \frac{N-1}{N}c - \tilde{V}_1(Z^*) \tag{6}$$

[Why does that make sense in the diagram? How much would the green triangle labelled PT grow if Z^* moved a little to the right? The rate of increase would be the height of the triangle, the distance between $(N-1)/N$ and $\tilde{V}_1(Z^*)$.]

There also is a little sense to this expression (6) for the marginal pivot tax. Suppose that person 1 gets the government to provide a little more of the public good. How does that affect the other people? The added cost that they would have to pay (since person 1 only has to pay for her share $1/N$ of the public good) is $[(N-1)N]c$ per unit. The added benefit they say that they would get (added up over all the other people) is $\tilde{V}_1(Z^*)$. So the marginal pivot tax is the net harm person 1 would do to these other people, if she gets the public sector to expand, past the point where the marginal cost to these other people equals the marginal benefit they get.

Suppose now that person 1 were really clever, and could manipulate the system to get any level Z^* of the public good that she wanted. What would her total taxes be, from a slight increase in the level of public good provision? From equations (2) and (6), the change in her total tax (part 1 plus the pivot tax) would be

$$\frac{\partial T}{\partial Z^*} + \frac{\partial PT}{\partial Z^*} = \frac{c}{N} + \frac{N-1}{N}c - \tilde{V}_1(Z^*) \tag{7}$$

Equation (7) can be simplified to

$$\frac{\partial T}{\partial Z^*} + \frac{\partial PT}{\partial Z^*} = c - \tilde{V}_1(Z^*) \tag{8}$$

Notice that if $Z^* > \tilde{Z}_1$, person 1 would have to pay more tax if she could somehow get the government to provide more of the public good.

But she also does get some benefit from the public good. She **doesn't** want simply to minimize her taxes : she wants to get the most benefit for the least taxes.

So let

$$p^1(Z)$$

denote person 1's **true** marginal willingness to pay for a little more of the public good. Only she knows that. But it does represent what she really does think a little more of the public good is worth. If $p^1(Z)$ is greater than the marginal taxes she would have to pay, then she would like to see the public good provision expanded. $p^1(Z)$ represents the benefit to her of a little more of the public good. Given the complicated tax rules — and given everyone else's announced benefits — the right side of equation (8) represents the marginal cost to her of a little more of the public good.

So if she could completely manipulate the system, and get any level Z^* of the public good that she wanted, then she would want a level Z^* such that her true marginal benefit equalled the increase in her taxes :

$$p^1(Z^*) = c - \tilde{V}_1(Z^*) \tag{9}$$

But the government has announced already what it will do to determine the public good level that it provides. It's using equation (3), which I can re–write as

$$v_1(Z^*) = c - \tilde{V}_1(Z^*) \tag{10}$$

Now look at equations (9) and (10). Equation (9) is the level of public good provision she wants, given this complicated tax rule. Equation (10) is the public good provision she will get, given that the government uses the Samuelson rule applied to people's announced demand curves. She can make what she gets into what she wants pretty simply : as long as $v_1(Z^*) = p^1(Z^*)$, then equations (9) and (10) are the same.

So she can get the best level of public good provision, from her selfish perspective, simply by telling the truth. Announcing a demand schedule

$$v_1(Z) = p^1(Z)$$

will guarantee that the solutions to equations (9) and (10) are the same, whatever the other people do. In the language of game theory, given these tax rules, announcing her true demand schedule is a **dominant strategy** to the game played by the taxpayers.

So, if people are clever and selfish, the government can get them to reveal voluntarily their willingness to pay for the public good, if it uses the Samuelson rule to provide the public good, and if it sticks to a policy of making each person's taxes equal the two parts, $T + PT$.

Not only does this mechanism get people to tell the truth, and provide the efficient quantity of the public good, it also ensures that there is enough tax revenue to pay for the public good. If we add up the first part of the taxes, over all people, the revenue sums up to cZ^*, exactly the cost of the public good. Then there are the pivot taxes PT. These must be non–negative : if $Z^* > \tilde{Z}_1$, then $[(N-1)N]c > \tilde{V}_1(Z)$ for $\tilde{Z}_1 < Z < Z^*$, so the pivot tax triangle has positive area. But if person 1 announces a low demand, and $\tilde{Z}_1 > Z^*$, then $\tilde{V}_1(Z) > [(N-1)/N]c$ for $Z^* < Z < \tilde{Z}_1$, so that the pivot tax triangle defined by expression (5) (and illustrated in figure 2) is again positive : person 1 is taxed for reducing public good provision, when other people value the marginal units of the public good more highly than the share of the cost that the have to pay.