Pure Public Goods : c A Numerical Example

Suppose that the utility function of person 1, as a function of her consumption x_1 of a pure private good, and z_1 of a pure public good, could be written

$$U^1(x_1, z_1) = 2\ln x_1 + \ln z_1$$

(where "$\ln x_1$" means the natural logarithm of x_1) and that the utility function of person 2 could be written (as a function of his consumption x_2 of the private good and z_2 of the public good)

$$U^2(x_2, z_2) = \ln x_2 + 2\ln z_2$$

These are both examples of **Cobb–Douglas** utility functions [1] . The two people also have different preferences : relative to person 1, person 2 has a stronger taste for the public good and a weaker taste for the private good.

Suppose that the equation of the economy's production possibility frontier [2] is

$$X = 120 - Z$$

In this example, the production possibility frontier is a straight line, with a (constant) slope of -1. The fact that the production possibility frontier [3] is a straight line means that the MRT — which is just the slope of the PPF — is a constant, which here equals 1.

With the preferences given above, the marginal rates of substitution of the two people, the ratios of the marginal utilities, are

$$MRS_1 = \frac{MU_z^1}{MU_x^1} = \frac{x_1}{2z_1}$$

$$MRS_2 = \frac{MU_z^2}{MU_x^2} = \frac{2x_2}{z_2}$$

Any efficient allocation must obey the these three conditions :

(1) everyone consumes the pure public good : $z_1 = z_2 = Z$

(2) the allocation is on the production possibility frontier : $x_1 + x_2 + Z = 120$

(3) the Samuelson condition : $MRS_1 + MRS_2 = 1$

In this example, when condition (1) is used to substitute for z_1 and z_2, the Samuelson condition becomes

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[1] see, for example, the appendix to chapter 4 of Varian's *Intermediate Microeconomics*, 8th edition

[2] see, for example, chapter 32 of Varian's text

[3] sometimes called the "production possibility **curve**

$$\frac{x_1}{2Z} + \frac{2x_2}{Z} = 1 \qquad (eg1)$$

Now any allocation (x_1, x_2, Z) which satisfies the Samuelson condition, and the feasibility constraint (2), is Pareto optimal.

For example, each of the following three allocations is an efficient allocation :

i : $x_1 = 0$; $x_2 = 40$; $z_1 = z_2 = Z = 80$

ii : $x_1 = 20$; $x_2 = 30$; $z_1 = z_2 = 70$

iii : $x_1 = 60$; $x_2 = 10$; $z_1 = z_2 = 50$

In other words, at least in this example, there is more than one efficient allocation, and more than one efficient level of public good provision.

Equation $(eg1)$ can be written

$$x_1 + 4x_2 = 2Z \qquad (eg2)$$

Any allocation (x_1, x_2, Z), in which all the consumption levels are non–negative, and which satisfies the optimality condition $(eg2)$ and the feasibility condition

$$x_1 + x_2 + Z = 120 \qquad (eg3)$$

will be efficient.

Since equation $(eg3)$ implies that $x_2 = 120 - x_1 - Z$, we can substitute for x_2 in equation $(eg2)$ to get

$$x_1 + 4(120 - x_1 - Z) = 2Z$$

or

$$Z = 80 - \frac{x_1}{2} \qquad (eg4)$$

In this example, equation $(eg4)$ completely describes all the efficient allocations. Take any $x_1 \geq 0$. Then calculate Z from equation $(eg4)$, and then x_2 from equation $(eg3)$: as long as all three numbers are non–negative, we have an efficient allocation.

So there are **many** efficient allocations in this example : **each** of them satisfies the Samuelson condition $(eg2)$ (as well as the feasibility condition $(eg3)$. The fact that there are many efficient allocations should not be surprising. Consider the problem of finding efficient allocations in an economy with only private goods, as in AP/ECON 2350 (for example, as in the Edgeworth box diagram) [4]. There will, in general, be many efficient allocations, some better for person #1, and some better for person #2. Notice in this example, as we increase x_1, we **decrease** Z,and decrease x_2 as well. That is because, in this example, person #1 has the relatively weaker taste for the pure public good. If the welfare function [5] were to give more importance to person #1, how would we

[4] as, for instance, in chapter 31 of Varian's text

[5] as in chapter 33 of Varian's text

make her better off, and person #2 worse off? First of all, we would give more of the private good to person #1. But we would also choose to produce more of the pure private good : giving more weight to person #1 in the welfare function means deciding on a production plan for the economy which is closer to her preferred plan, and she wants more of the private good and less of the public good.

Aside : A Particular Welfare Function

Consider the maximization of the welfare function

$$W(U^1, U^2) = aU^1 + U^2$$

where a is some positive constant. The higher is a, the more the welfare function gives importance to person #1. In this case,

$$\frac{\partial W}{\partial U^1} \equiv W_1 = a$$
$$\frac{\partial W}{\partial U^2} \equiv W_2 = 1$$

If you go back to the derivation of the efficiency conditions in the previous note, the first–order conditions for the social planner's optimization were

$$W_1 U_x^1 = \lambda \tag{$x1$}$$

$$W_2 U_x^2 = \lambda \tag{$x2$}$$

$$W_1 U_z^1 + W_2 U_z^2 = -\lambda F'(Z) \tag{Z}$$

In this example

$$U_x^1 = \frac{2}{x_1}$$

so that equation $(x1)$ implies that

$$\lambda = \frac{2a}{x_1} \tag{$swf1$}$$

Also

$$U_z^1 = \frac{1}{Z}$$
$$U_z^2 = \frac{2}{Z}$$

so that equation (Z) implies that

$$\frac{a}{Z} + \frac{2}{Z} = \lambda \tag{$swf2$}$$

Combining equations $(swf1)$ and $(swf2)$,

$$\frac{2+a}{Z} = \frac{2a}{x_1}$$

or

$$Z = \frac{2+a}{2a} x_1 \tag{$swf3$}$$

Finally, plugging equation ($swf3$) into the efficiency condition($eg4$), we get

$$\frac{2+a}{2a} x_1 = 80 - \frac{x_1}{2}$$

or

$$x_1 = \frac{a}{a+1} 80 \tag{$swf4$}$$

which then imply

$$Z = \frac{2+a}{2(a+1)} 80 \tag{$swf5$}$$

and

$$x_2 = \frac{40}{a+1} \tag{$swf6$}$$

Equations ($swf4$), ($swf5$), and ($swf6$) define completely the welfare–maximizing solution. For any positive level of a, the relative weight on person #1's well–being, these three equations define a feasible allocation which satisfies the Samuelson condition. For any positive level of a, the allocation defined by equations ($swf4$), ($swf5$) and ($swf6$) is efficient. As the weight a on person #1's well–being goes up.x_1 increases, and x_2 and Z decrease.

Left to the Reader

What would happen if person 1 had a stronger taste than person 2 for the public good, for example if $U^1 = \ln x_1 + 3 \ln z_1$ and $U^2 = \ln x_2 + \ln z_2$?

What would happen in both people had the same taste for the public good, for example if $U^1 = 4 \ln x_1 + \ln z_1$ and $U^2 = 4 \ln x_2 + \ln z_2$? Would there still many efficient allocations? Would there still be many efficient levels of public good provision?

(tricky) Here's another example in which both people have the same taste for the public good — but preferences are not Cobb–Douglas : $U^1 = 200\sqrt{x_1} + z_1$ and $U^2 = 200\sqrt{x_2} + z_2$

Adding up the Demand Curves Vertically

Returning to the original example, in which

$$U^1(x_1, z_1) = 2 \ln x_1 + \ln z_1$$

$$U^2(x_2, z_2) = \ln x_2 + 2 \ln z_2$$

recall that the demand functions of a person with Cobb–Douglas preferences

$$U(x, z) = a \ln x + b \ln z$$

are [6]

$$x^D = \frac{a}{a+b} \frac{M}{P_x}$$

$$z^D = \frac{b}{a+b} \frac{M}{P_z}$$

where P_x and P_z are the prices of the goods, and M the person's income.

Plugging into the example, the two people's demand curves for the pure public goods have the equations

$$z_1^D = \frac{1}{3} \frac{M_1}{P_z}$$

$$z_2^D = \frac{2}{3} \frac{M_2}{P_z}$$

(These demand curves confirm an allegation made at the beginning of this section, that person 2 has a stronger taste for the pure public good than does person 1. The demand curve for person 2 is above and to the right of the demand curve of person 1.)

To add up the demand curves vertically, these equations must be expressed in "inverse demand" format, showing how much each person is willing to pay as a function of the quantity she or he consumes of the pure public good.

Since each person will consume the same amount of the pure public good in an efficient allocation (if the good is indeed a "good"), the quantity each person consumes is just Z, the quantity provided of the pure public good. The height of each person's demand curve is the price she or he is willing to pay for a little more of the pure public good, which might be different for different people. Let P_z^1 denote how much person 1 is willing to pay for a little more of the public good, and P_z^2 how much person 2 is willing to pay. Then the equations for the demand curves can be written as

$$Z = \frac{1}{3} \frac{M_1}{P_z^1}$$

$$Z = \frac{2}{3} \frac{M_2}{P_z^2}$$

which can be re–arranged to express each person's willingness to pay as a function of the quantity provided of the public good :

[6] see the appendix to chapter 5 of Varian's text

$$P_z^1 = \frac{1}{3}\frac{M_1}{Z}$$

$$P_z^2 = \frac{2}{3}\frac{M_2}{Z}$$

Therefore, the vertical sum of the demand curves is

$$P_z^1 + P_z^2 = \frac{1}{Z}[\frac{1}{3}M_1 + \frac{2}{3}M_2]$$

This sum should be equal to the cost of the pure public good, which is equal to 1 in this example. (Recall that the production possibility curve has a constant slope of -1 in this example.)

So what is "the" level Z of public good provision Z for which the sum of the heights of the demand curves equals the marginal cost of the pure public good? The level depends on what are people's incomes. This fact should not be too surprising : changes in income will usually shift demand curves. If person 1's income goes up by a dollar, and person 2's goes down by a dollar, then person 1's demand curve shifts out, and person 2's shifts in. But they don't shift by the same amount in this case. A dollar increase in M_1, accompanied by a dollar decrease in M_2, will actually lower $P_z^1 + P_z^2$ by $1/3Z$, as the above equation shows.

In this example, the total income of the two people would be 120, if there were no public good provided (since the equation of the production possibility frontier is $X + Z = 120$, and since the price of the private good X is 1). The condition that the vertical sum of the demand curves equal the marginal cost is

$$\frac{1}{Z}[\frac{1}{3}M_1 + \frac{2}{3}M_2] = 1$$

or

$$M_1 + 2M_2 = 3Z$$

As the income distribution moves from person 1 having all the money ($M_1 = 120$, $M_2 = 0$) to person 2 having all the money ($M_1 = 0$, $M_2 = 120$), the efficient quantity to provide of the public good increases from 40 to 80.