Many pure public goods are provided not by the government, nor by for–profit private firms, but by voluntary subscriptions. (Examples : charities, neighbourhood improvements, non–profit broadcasting.) What would be the outcome if a pure public good were financed exclusively by voluntary donations?

Here it will be assumed that individuals act purely in their own self–interest, and care only about their own consumption bundles. This is a standard economic assumption, but rules out a lot of phenomena which may be relevant. The assumption means that when a person contemplates how much to give to some campaign to fund public broadcasting, all that she cares about is the amount of public broadcasting that she'll get to watch, and the amount of her own money that she will have left for spending on other (private) goods. That means that she don't care about the size of her own donation, except in its effect on the total funding (and in its effect on how much money she has left to spend on herself). In practice, many people care about how much they themselves donate, for its own sake : maybe they care about their reputation for generosity, or just get a "warm glow" from giving. For example, why do people donate to disaster relief? Certainly, they donate because they get a benefit from knowing that survivors of a hurricane or tsunami will be aided. The better off the survivors are, the better off donors feel. But donors, typically, do care about who gives the money. A donor gets more of a "good feeling" from contributing \$100 herself, than from knowing that someone else contributed \$100.

The model presented here will neglect this important aspect of many activities supported by voluntary contributions : that people care not just about how much is raised, but about how much they themselves contributed.

Under this assumption of self interest, person i's preferences can be represented by a utility function

$$U^i(x_i, Z)$$

where x_i is her spending on private goods for herself, and Z is the total quantity she gets to consume of the public good.

If M_i is person i's income, then the income she will have available for spending on private goods is her total income, minus what she contributes to public good provision. If ζ_i is what she contributes to public good provision, then

$$x_i = M_i - \zeta_i$$

Let the unit cost of the public good be c, so that the total cost of Z units of the public good is cZ, a cost which must be financed by people's voluntary contributions

$$cZ = \zeta_1 + \zeta_2 + \cdots + \zeta_N$$

if there are N people who are potential contributors.

These definitions mean that person 1's utility, for example, equals

$$U^1(M_1 - \zeta_1, \frac{1}{c}[\zeta_1 + \zeta_2 + \cdots + \zeta_N])$$

Notice that increasing her contribution ζ_1 has two effects : it lowers the amount of money she has available for spending, but it increases the amount of the public good available.

Overall, the effect of a change in ζ_1 on her utility is

$$\frac{\partial U^1}{\partial \zeta_1} = \frac{\partial U^1}{\partial Z}\frac{1}{c} - \frac{\partial U^1}{\partial x_1}$$

The first term on the right side of the equation above is the benefit from the increase her contribution provides in the level of public expenditure Z ; the second term is the harm from the reduction in her spending on her own private good consumption. From the definition of her marginal rate of substitution

$$MRS^1 \equiv \frac{\partial U^1}{\partial Z} / \frac{\partial U^1}{\partial x_1}$$

this overall change in utility is

$$\frac{\partial U^1}{\partial \zeta_1} = \frac{\partial U^1}{\partial x_1}[\frac{1}{c}MRS^1 - 1]$$

Acting in her own self–interest, person 1 will want to choose a level of contributions ζ_1 to make her utility as high as possible. As usual, maximization of U^1 means setting the derivative of U^1 with respect to ζ_1 equal to 0, or

$$MRS^1 = c \tag{$*$}$$

Why does this condition make sense? The benefit to the person of a little more of the public good is her willingness to pay for the public good, her MRS. If provision requires voluntary contributions, the cost to her of having one more unit of the public good is c : she can increase the total provision of the public good only by providing the money herself, and it costs c to provide one more unit of the public good.

But there is an important complication here. Typically, people can choose to contribute to a voluntary campaign, or not to contribute. They can't take money out of the campaign funds. There is a restriction on contributions, that $\zeta_1 \geq 0$. When $\zeta_1 = 0$, the net marginal benefit of donating a little is proportional to

$$\frac{\partial U^1}{\partial x_1}[MRS^1(M_1, \zeta_2 + \zeta_3 + \cdots + \zeta_N) - c]$$

The willingness to pay MRS^1 depends on the level of x_1 and Z. It decreases with Z ; the demand curve for the public good slopes down. If other people have been generous, so that $\zeta_2 + \zeta_3 + \cdots + \zeta_N$ is large, then it may be the case that $MRS^1 < c$ even when $\zeta_1 = 0$. In that case, the person's preferred level of contribution is 0, since she can't make negative contributions.

So what should the selfish prospective donor do? If

$$MRS^1(M_1, \zeta_2 + \zeta_3 + \cdots + \zeta_N) < c$$

she should donate nothing. Otherwise, she should donate an amount $\zeta_1 > 0$ such that

$$MRS^1(M_1 - \zeta_1, \zeta_1 + \zeta_2 + \zeta_3 + \cdots + \zeta_N) = c$$

Her choice of how much to give depends on what everyone else has given : other people giving more tends to raise Z, lowering her MRS, and making her give less.

What is the likely outcome of the voluntary provision, if people act selfishly, and care only about their x_i's and Z? Each person will behave like person 1, choosing a donation level to maximize $U^i(M_i - \zeta_i, \zeta_1 + \zeta_2 + \cdots + \zeta_N)$, taking as given the donations made by the other people. Each person's best strategy (how much to give) depends on what the other people do.

There is a standard economic solution to problems in which people act selfishly, and in which each person's best choice of action depends on what other people do. This solution is the **Nash equilibrium**. (See chapter 10 of Nicholson, or chapter 28 of Varian, for example.)

The levels of contributions $(\zeta_1, \zeta_2, \ldots, \zeta_N)$ constitute a Nash equilibrium if for each person i, ζ_i is her best choice of donation, given what everyone else is doing. In other words, $(\zeta_1, \zeta_2, \ldots, \zeta_N)$ is a Nash equilibrium vector of contributions if no person would want to change her or his ζ_i, given what the other people are doing.

The analysis of person 1's problem above shows that the Nash equilibrium outcome will be the following pattern of contributions : some people will donate nothing, and others will make contributions. If $MRS^i(M_i, Z) < c$ for person i, then she will be one of the people who choose to donate nothing. If $MRS^i(M_i - \zeta_i, Z) = c$ — with $\zeta_i > 0$, then person i is one of the people who make a positive donation .

A neat feature of this voluntary contribution model is that, for a given distribution of income $(M_1, M_2, \ldots, M_N)$ there is **exactly one** Nash equilibrium pattern of contributions $(\zeta_1, \zeta_2, \ldots, \zeta_N)$. (Warning : this uniqueness of equilibrium applies for voluntary contributions to a public good of variable quantity ; it does not apply for voluntary contributions to an "all–or–nothing" project.). So, if we believe that people care only about their x_i and Z, the model makes a clear prediction of the outcome.

The main feature of the outcome when people donate voluntarily (and are selfish) is that "too little" of the public good is provided. Efficient allocations when there is a public good should obey the Samuelson condition $MRS^1 + MRS^2 + \cdots + MRS^N = c$. Here, if everyone chose to contribute a positive amount, then each $MRS^i = c$, so that

$$MRS^1 + MRS^2 + \cdots + MRS^N = Nc$$

Since demand curves slope down, the fact that the sum of people's willingness to pay MRS^i for the public good exceeds c means that the quantity of the public good is too small. If some people chose not to contribute to the public good, then $c > MRS^i > 0$ for some people, so that

$$MRS^1 + MRS^2 + \cdots + MRS^N > nc$$

where n is the number of people ($n \leq N$) who actually choose to contribute. As long as more than one person contributes, the sum of the MRS's is greater than c.

The problem here with self–interested behaviour by people is that each person's contribution benefits not only herself, but everyone else in the group. In competitive markets for private goods, people acting in their own self–interest leads to an equilibrium which is **Pareto optimal** : there is no way of making everyone better off without making someone worse off. In contrast, the Nash equilibrium to the voluntary contribution game is not Pareto optimal. There are other outcomes which could make everyone better off.

For example, starting from the Nash equilibrium, suppose that every person making positive contributions were to increase her or his contribution by the same (small) amount from the equilibrium levels. If there are n people who were making positive contributions, and each of them increased their contributions by one dollar, then the total quantity of the public good would increase by n/c, since one dollar buys $1/c$ units more of the public good. The change in person 1's utility, if she was making a positive contribution, would be

$$\Delta U^1 = \frac{n}{c} MU_z^1 - MU_x^1 = \frac{1}{c} MU_x^1 [nMRS^1 - c]$$

since the quantity of the public good would go up by n/c, and the amount of money she had or private consumption would fall by 1. At the Nash equilibrium, her contribution was determined by the condition $MRS^1 = c$. So $nMRS^1$ will be much greater than c, if there are several people choosing to contribute in the Nash equilibrium. In other words, it is possible for all the people to be made better off, compared to the Nash equilibrium, if they could somehow coordinate an increase in their contributions. Repeating, this stands in strong contrast to the situation in markets for private goods : there, no coordinated action could improve on the equilibrium for everyone.

The failure of voluntary contribution to lead to a Pareto optimal allocation — when people behave non–cooperatively — suggests there may be a role for the government in providing the public good. Governments can force people to "contribute" more than they would under voluntary provision. If the government somehow knows people's preferences, it can improve on the outcome under voluntary provision.

Of course, there are many public goods which are provided (at least partially) through voluntary contributions. Fund–raisers for this sort of good seem to be aware of the under–provision. In the model presented above, people simply decided how much to contribute, taking each other person's contributions as given. Fund–raising campaigns often attempt to tie one person's contribution to other people's : in effect, by trying to arrange a campaign in which "I'll give more if you'll give more", a fund–raising campaign for a public good may be attempting the coordinated increase in contributions above the Nash equilibrium described above.

There is one other peculiarity of the Nash equilibrium to the voluntary contribution game, which might be worth mentioning. Income transfers between contributors are completely "crowded out" here. That is, if person 1's income were to go up by a dollar, and person 2's were to go down by a dollar, then there would be a new Nash equilibrium to the voluntary contribution game —

since people's incomes, which affect their demands, would have changed. In that new equilibrium, person 1's contributions to the public good will have increased by \$1, and person 2's will have decreased by \$1 — meaning the income transfer has not actually made person 1 better off or person 2 worse off.

Mathematically, why this happens is that there is a unique equilibrium to the voluntary contribution game. If there are only 2 people, and if both choose to make positive contributions, then the equilibrium can be described by three equations :

$$MRS^1(x_1, Z) = c$$

$$MRS^2(x_2, Z) = c$$

$$x_1 + x_2 + cZ = A$$

where A is the total income of the two people (available for spending on either the private good or the private good). These three equations define the equilibrium consumption levels x_1, x_2 and Z. If person 1 has income M_1 and person 2 has income M_2 (with $M_1 + M_2 = A$), then this equilibrium allocation will be achieved with

$$\zeta_1 = M_1 - x_1$$

$$\zeta_2 = M_2 - x_2$$

Increase M_1 by a dollar, and decrease M_2 by a dollar, then the same three equations will now be satisfied by

$$\zeta_1' = M_1 + 1 - x_1$$

$$\zeta_2' = M_2 - 1 - x_2$$

so that income transfers have no effects, once people's equilibrium contributions have adjusted.

Of course, in reality, it does not appear as if changes in people's incomes are completely offset by changes in their contributions towards voluntarily provided public goods, further evidence that people do not behave exactly as assumed in this model.