*Q*1. If person 1's demand function for a public good could be written

$$z_1 = \frac{Y_1}{3p_1}$$

and person 2's demand function for the public good could be written

$$z_2 = \frac{2Y_2}{3p_2}$$

where z_i is the quantity demanded of the public good by person i, as a function of her income Y_i and of the personalized ("Lindahl") price p_i she has to pay for the public good, then what is the quantity which should be provided of the public good — as a function of the people's incomes — when Lindahl ("benefit") taxation is used to finance the public good, if the cost of 1 unit of the public good is \$1?

*A*1. The Lindahl allocation is the allocation for which the sum of people's "Lindahl prices" equals the marginal cost of the public good. Person i's Lindahl price p_i is her marginal willingness to pay (in dollars) for a little more of the public good, as a function of the quantity Z of the public good provided, and of her income Y_i. So if $p_i(Z, Y_i)$ is the height of person i's demand curve for the public good, as a function of the quantity Z provided of the public good and of her income Y_i, the quantity Z provided of the public good in the Lindahl solution is the quantity Z such that

$$p_1(Z, Y_1) + p_2(Z, Y_2) = c \tag{1-1}$$

where c is the marginal cost of the public good.

Since person 1's demand curve for the public good has the equation $z_1 = \frac{Y_1}{3p_1}$, her inverse demand curve, expressing the price she is willing to pay, as a function of the quantity Z she consumes, is

$$p_1(Z, Y_1) = \frac{Y_1}{3Z} \tag{1-2}$$

Similarly, since person 2's demand function is $z_2 = \frac{2Y_2}{3p_2}$, her inverse demand function is

$$p_2(Z, Y_2) = \frac{2Y_2}{3Z} \tag{1-3}$$

Here the marginal cost of the public good is $c = 1$, so that (using equations $(1-2)$ and $(1-3)$) the condition $(1-1)$ defining the quantity of the public good in the Lindahl solution is

$$\frac{Y_1}{3Z} + \frac{2Y_2}{3Z} = 1 \tag{1-4}$$

or

$$Z = \frac{1}{3}(Y_1 + 2Y_2) \tag{1-5}$$

Equation $(1-5)$ defines the quantity Z of the public good, as a function of the people's incomes.

[In the Lindahl solution, the equilibrium prices paid by each person are

$$p_1 = \frac{Y_1}{Y_1 + 2Y_2} \qquad (1-6)$$

$$p_2 = \frac{2Y_2}{Y_1 + 2Y_2} \qquad (1-7)$$

resulting in taxes $T_i = p_i Z$ of

$$T_1 = \frac{Y_1}{3}$$

$$T_2 = \frac{2Y_2}{3}$$

imposed on the two people.]

*Q*2. Briefly describe a mechanism which will induce people to reveal truthfully their inverse demand curves $p_i(Z)$ for a public good (where $p_i(Z)$ denotes person i's marginal willingness to pay for a little more of the public good, as a function of the quantity Z provided of the public good).

*A*2. One example of this sort of mechanism is the one described in section $2(b)$ of the lecture notes. Let $a_i(Z)$ be what person i **announces** is her inverse demand function for the public good (so that $a_i(Z)$ is what person i says is her willingness to pay, in dollars for a little more of the public good, when the quantity available is Z.)

Suppose that there are N people in the country, and that the cost of providing Z units of the public good is cZ. Then the mechanism consists of the following rules for public good provision and taxes, depending on what people announce as their inverse demand functions for the public good.

(1) The quantity Z^* provided of the public good is the quantity Z^* for which

$$a_1(Z^*) + a_2(Z^*) + \cdots + a_N(Z^*) = c \qquad (2-1)$$

(2) Each person pays an equal share $\frac{cZ^*}{N}$ of the cost of the public good.

(3) In addition, each person pays an additional "pivot tax". For person 1, her pivot tax is the area between the sum of everyone else's announced inverse demand functions $A_1(Z) \equiv a_2(Z) + a_3(Z) + \cdots + a_N(Z)$, and a horizontal line of height $\frac{N-1}{N}c$, between the quantities Z_1 and Z^*, where Z_1 is defined by the condition

$$a_2(Z_1) + a_3(Z_1) + \cdots + a_N(Z_1) = \frac{N-1}{N}c \qquad (2-2)$$

That is, person 1's pivot tax $T_1(Z_)$ is

$$T_1(Z) = \int_{Z_1}^{Z^*} [\frac{N-1}{N}c - a_2(Z) - a_3(Z) - \cdots - a_N(Z)]dZ \qquad (2-3)$$

where Z_1 is defined by $(2-2)$ above.

[The pivot tax for other people (other than person #1) is defined analogously, with Z_i defined as the level of public good provision where the sum of the announced demand curves of everyone else but person $\#i$ cross a line of height $\frac{N-1}{N}c$.]

If the government asks people to announce their willingness to pay for the public good, and commits to using rules (1), (2) and (3) to decide the quantity Z of the public good, and the taxes paid by each person, then how is person #1 affected by a slight increase in the quantity of Z^* of the public good provided? She benefits $p_1(Z^*)$ (her true marginal willingness to pay) from the increase, but her taxes go up by $c/N + \frac{\partial T_1}{\partial Z^*}$.

So her net gain from a slight increase in Z^* is

$$p_1(Z^*) - \frac{c}{N} - \frac{\partial T_1}{\partial Z^*} \qquad (2-4)$$

Ideally, she would like to give an answer which increases the quantity provided of the public good up to the point at which expression $(2-4)$ equals zero : at this point the marginal benefit to her of a little more of the public good just equals the added taxes she must pay for the increase.

From expression $(2-3)$,

$$\frac{\partial T_1}{\partial Z^*} = \frac{N-1}{N}c - a_2(Z^*) - a_3(Z^*) - \cdots - a_N(Z^*) \qquad (2-5)$$

so that she wants a level Z^* of the public good for which

$$p_1(Z^*) - \frac{c}{N} - \frac{N-1}{N}c + a_2(Z^*) + a^3(Z^*) + \cdots + a_N(Z^*) = 0 \qquad (2-6)$$

or

$$p_1(Z^*) + a_2(Z^*) + a_3(Z^*) + \cdots + a_N(Z^*) = c \qquad (2-7)$$

But — given that equation $(2-1)$ above is used to choose the quantity Z^* of the public good — she can guarantee that her optimality condition $(2-7)$ will hold, simply by setting $a_1(Z) = p_1(Z)$ for any quantity Z of the public good, that is by announcing truthfully her inverse demand function for the public good.

*Q*3. Suppose there is some input Z with the following properties : increases in firm 1's purchases Z_1 of the input lead to increased profits for firm 2, and increases in firm 2's purchases Z_2 of the input lead to increased profits for firm 1.

Is the equilibrium allocation efficient, when each firm chooses its own input quantities so as to maximize its own profit?

Explain briefly.

*A*3. The equilibrium allocation will not be efficient : each firm i will choose **too low** a level Z_i of its own purchases of the externality–causing input.

Formally, the joint profits of the two firms are

$$\pi_1 + \pi_2 = p_1 F^1(Z_1, Z_2) + p_2 F^2(Z_2, Z_1) - w_Z(Z_1 + Z_2) \tag{3-1}$$

where p_i is the price of firm i's output, F^i is the quantity of output produced by firm i (which depends on both its own use of input Z, and on the use of the input by the other firm), and where w_Z is the price of the input. Maximizing expression $(3-1)$ with respect to the input quantities Z_1 and Z_2 implies that

$$p_1 \frac{\partial F^1}{\partial Z_1} + p_2 \frac{\partial F^2}{\partial Z_1} = w_Z \tag{3-2}$$

$$p_2 \frac{\partial F^2}{\partial Z_2} + p_1 \frac{\partial F^1}{\partial Z_2} = w_Z \tag{3-3}$$

Acting on its own, firm 1 would look to maximize its own profits

$$\pi_1 = p_1 F^1(Z_1, Z_2) - w_Z Z_1 \tag{3-4}$$

with respect to its own choice Z_1 of input purchases. Maximizing $(3-4)$ with respect to Z_1 implies that

$$p_1 \frac{\partial F^1}{\partial Z_1} = w_Z \tag{3-5}$$

So equation $(3-2)$ can be written

$$MB_1^1 + MB_1^2 = w_Z \tag{3-6}$$

where MB_j^i is the — positive – marginal benefit of a slight increase in Z_j on the profits of firm i. Since firm 1, acting unilaterally, chooses a level of input such that

$$MB_1^1 = w_Z \tag{3-7}$$

it is ignoring the positive marginal benefits MB_1^2 to the other firm in making its decision, and will choose too low a level of input Z.

Similarly, firm 2 chooses a level of Z_2 such that

$$MB_2^2 = w_Z \tag{3-8}$$

when efficiency requires

$$MB_2^2 + MB_2^1 = w_Z \tag{3-9}$$

so that it chooses too low a level of Z_2 when it ignores the positive benefits of Z_2 on π_1.

So, just because each firm's activity affects the other does not mean that the externalities somehow cancel each other out. And just because the externality is positive does not mean that it can be ignored.