AP/ECON 4080 Answers to Mid–term Exam February 2013

$Q1$. What are the conditions which must be satisfied for an allocation to be efficient in an economy in which there are many people, and 3 goods : 2 private goods and 1 public good?

$A1$. There are three efficiency conditions for this economy :

- The aggregate production plan must be feasible, and on the production possibility frontier.
- Each person's marginal rate of substitution between the two private goods must be the same, and must equal the marginal rate of transformation between those goods.
- The sum of the 3 people's marginal rates of substitution between the public good and one of the private goods, must equal the marginal rate of transformation between the public good and that private good.

Mathematically, if $X$ and $Y$ represent the aggregate quantities of the two private goods, and $Z$ represents the aggregate quantity of the public good, and if $x^i$, $y^i$ and $z^i$ represent person $i$'s consumption of the two private goods and of the public good, then we must have

- $X = x^1 + x^2 + \cdots + x^I$ and $Y = y^1 + y^2 + \cdots + y^I$ if there are $I$ people in the economy
- $z^1 = z^2 = \cdots = z^I = Z$
- $X = F(Y, Z)$, which is the production possibility frontier's equation, stating the maximum amount $X$ of the first private good which can be produced, if $Y$ units of the other private good are produced, and $Z$ units of the public goods
- if the preferences of person $i$ can represented by a utility function $U^i(x^i, y^i, z^i)$, then

$$\frac{U_y^1}{U_x^1} = \frac{U_y^2}{U_x^2} = \cdots = \frac{U_y^I}{U_x^I} = F_Y$$

  where $U_x^i$ and $U_z^i$ are the marginal utilities with respect to $x^i$ and $y^i$ respectively, and $F_Y$ is the partial derivative of $(Y, Z)$ with respect to $Y$

- $$\frac{U_z^1}{U_x^1} + \frac{U_z^2}{U_x^2} + \cdots + \frac{U_z^I}{U_x^I} = F_Z$$

[It also must be true that $\frac{U_z^1}{U_y^1} + \frac{U_z^2}{U_y^2} + \cdots + \frac{U_z^I}{U_y^I} = \frac{F_Z}{F_Y}$, but this must be true if the last two efficiency conditions hold].

*Q*2. How much tax revenue would be collected by the following "pivot tax" mechanism, if each person tries to use the mechanism to make herself as well off as possible?

The indivisible ("all or nothing") public project costs \$5000. There are 5 people : each person knows how much she herself values the project (but nobody else knows her valuation). People #1, #2 and #3 each value the project at \$1400, and people #4 and #5 each value the project at \$500.

The rules of the tax are : the project will be undertaken if and only if the sum of the 5 people's announced valuations exceeds the cost of the project, \$5000. If the project is undertaken, each person will pay the same share, \$1000, of the cost. In addition, if any person is "pivotal" (that is, if her valuation alters the overall result), then she will have to pay a pivot tax, equal to the (absolute value of the) difference between the sum of everyone else's announced valuations and the sum of the shares of the cost (4000) which they must pay.

*A*2. The project will be undertaken here, if people report their true valuations, since the sum of the five people's valuations is $1400 + 1400 + 1400 + 500 + 500 = 5200 > 5000$.

Given the tax mechanism described here, reporting one's true valuation is a dominant strategy for each person, so they will report their valuations truthfully (if they understand the workings of the rules).

So the project gets undertaken, and each person pays a tax of \$5000 to pay for its cost.

In addition, anyone who is "pivotal" pays the extra pivot tax described in the question.

A person is pivotal if the average valuation of the **other four** people is less than \$1000 : without this person, the average valuation is below the threshold, but with this person the average value exceeds the threshold.

So people #1, #2 and #3 are all pivotal. For example, without person #1, the sum of the other 4 people's valuations is $1400 + 1400 + 500 + 500 = 3800 < 4000$, so that the average of the other 4 people's valuations is less than \$1000.

By the rules of the pivot tax, person #1 would pay a pivot tax of \$4000, minus the sum of everyone else's valuation, or $4000 - 3800 = 200$.

People #2 and #3 are in exactly the same situation, so that they too pay a pivot tax of \$200.

So, if people understand the mechanism, the project will be undertaken, and each of the three pivotal voters (#1, #2 and #3) pays a pivot tax of \$200. The total tax revenues collected are \$5000 to pay for the project, plus \$600 in (extra) pivot tax revenue.

*Q*3. Suppose that the output produced by firm #1 was an increasing function of the number of workers hired by firm #2 (as well as an increasing function of the number of workers hired by firm #1 itself), and that the output produced by firm #2 was an increasing function of the number of workers hired by firm #1 (as well as an increasing function of the number of workers hired by firm #2 itself).

Is there a need for government intervention to correct an externality in the situation just described? Explain.

*A*3. In this example, there are positive externalities. If they ignore each other, each firm will hire **too little** labour, since they take account only of the impact of the labour they hire on their own output, and not on the output of the other firm.

If labour is in perfectly elastic supply at a cost of $w$ per hour, and the level of firm 1's output can be written $Q_1 = F(L_1, L_2)$, then, ignoring the externality, firm 1 will hire labour up until the level $L_1^0$ at which

$$p_1 F_1 = w$$

where $p_1$ is the price of firm #1's output and $F_1$ is the derivative of $F$ with respect to $L_1$. But firm 2's output is $G(L_1, L_2)$, depending on both firms' labour inputs, so that the efficient level of $L_1^*$ is the level for which

$$p_1 F_1 + p_2 G_1 = w$$

Since $G_1 > 0$, $L_1^* > L_1^0$ : firm 1 hires an inefficiently low quantity of labour, if it considers only its own private benefit of increased labour use, rather than the overall social benefit.

Similarly, firm 2 would choose, on its own, to hire the quantity $L_2^0$ of labour for which

$$p_2 G_2 = w$$

when the efficient level $L_2^*$ of $L_2$ is the level for which

$$p_1 F_2 + p_2 G_2 = w$$

If firms cannot negotiate, or merge, government intervention may be required. The government could order both firms to increase their employment of labour (up to the efficient level at which the marginal increase from a firm's labour hiring on the sum of the value of both firms' outputs equals the wage per unit of labour). Or the government could subsidize both firms' use of labour : with the subsidy rate equaling the value of the marginal increase in the other firm's output.

However, if the firms merge, the new merged entity will take account of the externality from one branch of the new merged firm onto the other branch, and will hire the efficient quantity of labour for each branch.

Even if firms do not merge, if they can negotiate, they will be able to reach the efficient outcome without government intervention. Here the firms might agree to a deal in which firm 1 agrees to expand its labour hiring up to the overall efficient level, provided that firm 2 agrees to expand its labour hiring of to the overall efficient level.