AP/ECON 4080 : April 2014 : sketches of answers to part B (long answer) of final exam

*Q*1. What are all the Pareto optimal allocations in the following 2–person, 2–good economy? Good X is a pure private good, and good Z is a pure public good. The feasible production possibilities for the economy are those (X, Z) combinations for which

$$X + Z \leq 48$$

where X is aggregate production of the pure private good and Z is aggregate production of the pure public good. Person 1's preferences can be represented by the utility function

$$u^1(x_1, z_1) = 2 \log x_1 + \log z_1$$

and person 2's by the utility function

$$u^2(x_2, z_2) = \ln x_2 + 3 \ln z_2$$

where x_i is person i's consumption of the private good, and z_i is person i's consumption of the public good (and where "log" refers to the natural logarithm).

*A*1. Given the preferences, the marginal utilities of private and public good consumption for the two people are

$$MU_x^1 = \frac{2}{x_1} \qquad MU_z^1 = \frac{1}{z_1} \tag{1 − 1}$$

$$MU_x^2 = \frac{1}{x_1} \qquad MU_z^2 = \frac{3}{z_2} \tag{1 − 2}$$

From equations $(1-1)$ and $1-2)$, the two people's marginal rates of substitution between the public good and the private good are

$$MRS_{zx}^1 = \frac{MU_z^1}{MU_x^1} = \frac{x_1}{2z_1} \tag{1 − 3}$$

$$MRS_{zx}^2 = \frac{MU_z^2}{MU_x^2} = \frac{3x_2}{z_2} \tag{1 − 4}$$

For an allocation to be efficient, it must be true that $z_1 = z_2 = Z$ if good z is non–rivalrous, and that $MRS^1 + MRS^2 = MRT$ (the Samuelson condition). Since the production possibility curve here s a straight line, with equation $X + Z = 48$, therefore the MRT is a constant, 1. So equations $(1-3)$ and $(1-4)$ imply that the Samuelson condition can be written

$$\frac{x_1}{2Z} + \frac{3x_2}{Z} = 1 \tag{1 − 5}$$

or

$$x_1 + 6x_2 = 2Z \tag{1 − 6}$$

The production combination must also be on the production possibility frontier for the allocation to be efficient, so that, in this case, (x_1, x_2, Z) must satisfy the equation

$$x_1 + x_2 + Z = 48 \tag{1-7}$$

Substituting for Z from equation $(1-7)$ into equation $(1-6)$ yields

$$x_1 + 6x_2 = 2(48 - x_1 - x_2) \tag{1-8}$$

or

$$3x_1 + 8x_2 = 96 \tag{1-9}$$

Any (x_1, x_2) combination satisfying equation $(1-9)$, with $x_1 \geq 0$ and $x_2 \geq 0$ will be an efficient allocation (with $Z = 48 - x_1 - x_2$)).

So, for example, the allocations $(x_1, x_2, Z) = (0, 12, 36)$, $(x_1, x_2, Z) = (8, 9, 31)$, $(x_1, x_2, Z) = (24, 3, 21)$ and $(x_1, x_2, Z) = (32, 0, 16)$ are all efficient.

*Q*2. Suppose there is some public good, and the government is trying to find the efficient quantity to provide of this public good. The quantity of the public good can be varied, and the cost of one unit of the public good is some constant c.

The government chooses to ask each person to report her demand curve for this public good.

Describe a rule for determining the quantity of the public good, and the taxes paid by different people, so that each person would find it in her own interest to report truthfully her demand curve for the public good.

*A*2. This question refers to “partial equilibrium preference revelation mechanisms”. An example of such a mechanism is a rule which has three different parts :

(i) The level of public good provision will be Z^*, defined by the equation

$$\sum_{i=1}^{I} v_i(Z^*) = c \tag{2-1}$$

where $v_i(Z)$ is the demand curve reported by person i, and I is the number of people.

(ii) Each of the I people pays an equal share $\frac{cZ^*}{I}$ of the cost of the public good.

(iii) In addition, each person i pays an extra tax, defined below, based on how much the person's reported demand curve affected the outcome :

The extra tax is defined as

$$t_i = \int_{Z_i}^{Z^*} [\frac{I-1}{I}c - \sum_{j \neq i} v_j(Z)]dZ \tag{2-2}$$

where the term Z_i in equation $(2-2)$ is defined by

$$\sum_{j \neq i} v_j(Z_i) = \frac{I-1}{I}c \tag{2-3}$$

If the government announces a rule with these properties, then a selfish individual, who wants to maximize her benefits from the public good, minus her taxes

$$B_i(Z^*) - \frac{cZ^*}{I} - \int_{Z_i}^{Z^*} [\frac{I-1}{I}c - \sum_{j \neq i} v_j(Z)]dZ \tag{2-4}$$

where $B_i(Z)$ is the person's **true** total benefits from a level of public good provision of Z, will find her best selfish strategy is to report a demand function

$$v_i(Z) = B_i'(Z)$$

to the government. That is, she should report truthfully her marginal benefits from the public good.

*Q*3. If access to a common property resource can be controlled, by charging a price for the use of the resource, how should the price be set so as to maximize the net value of the resource?

*A*3. Since free entry to a common property resource leads to excessive exploitation of the resource, lowering the net value, a positive entry fee is needed to limit entry. To maximize the net value, the entry fee for each use of the resource should be set equal to to the value of the (negative) externality imposed by the user. That's the reduction in the net value of every other user.

Formally, suppose that N is the number of users of the common property resource, and $F(N)$ is the total output of each user. If $F'(N) < 0$ we have an externality problem : each new user increases N, which therefore will reduce the output $F(N)$ that every other user gets.

If p is the price of the output, and c is the cost of of entering, then the net return to each user is $pF(N) - c$. So the total net value of all the users is

$$V \equiv N(pF(N) - c) \tag{3-1}$$

Maximization of net value means finding the number N^* of users which maximizes expression $(3-1)$. Differentiating expression $(3-1)$,

$$V'(N) = pF(N) - c + pNF'(N) \tag{3-2}$$

The efficient number N^* of users is the value of N which makes expression $(3-2)$ equal 0.

An individual user will enter if and only if she can make a profit from entry. If an entry fee f is being charged, the user's profit is

$$pF(N) - c - f \tag{3-3}$$

In equilibrium, users will enter up to the point where the profit has been reduced to zero. So the level $\hat{N}$ of entry, when users are free to enter, is determined by

$$pF(\hat{N}) = c + f \tag{3-4}$$

Comparing expressions $(3-2)$ and $3-4)$, the fee f^* which makes $\hat{N}$ equal to N^* is

$$f^* = -pN^*F'(N^*) \tag{3-5}$$

Charging this entry fee to each user will lead to the net value of the common property resource being maximized. Note that the **reduction** in each other firm's profit from a slight increase in N is $-pF'(N)$, so that the fee f^* defined in expression $(3-4)$ is the total reduction in the profits of **all** the other users of a slight increase in N.

*Q*4. How many kilometres of highways would be built in the city described below, if all the residents of the city got to vote over all possible amounts of highway building?

The highway is to be financed by a proportional income tax. Each voter has the same preferences, represented by the utility function

$$U(x, H) = x + 3\sqrt{H}$$

where x is the person's after–tax income, and H is the number of kilometres of highway built in the city.

Highways cost \$1 million dollars per kilometre to build. There are 1 million people in the city. The average (mean) income in the city is \$80,000. The median income in the city is \$60,000.

*A*4. Since the highways are to be financed by a proportional income tax, it must be true that

$$tY = cH \tag{4-1}$$

if the city's government's budget is balanced, where t is the income tax rate in the city, Y is the total income, and c is the cost per kilometre of building highways. Now the total income in any jurisdiction is equal to the average income times the population

$$Y = \bar{y}P \tag{4-2}$$

In this question, $c = 1000000$ and $P = 1000000$. So equations $(4-1)$ and $(4-2)$ become

$$(80,000)t = H \tag{4-3}$$

Equation $(4-3)$ relates the benefit a voter gets from more construction H to the tax rate t which she must pay to finance the highways.

A person's after tax income x is

$$x = (1-t)y \tag{4-4}$$

if her income is y, so that her utility will be

$$u = (1-t)y + 3\sqrt{H} \tag{4-5}$$

if the tax rate is t and if H kilometres of highways are built. Substituting from $(4-3)$ for t,

$$u(H;y) = y - \frac{y}{80000}H + 3\sqrt{H} \qquad (4-6)$$

Equation $(4-7)$ expresses the voter's utility as a function of the level H of highway construction. It is a concave function of H :

$$u_H(H;y)) = \frac{3}{2}\frac{1}{\sqrt{H}} - \frac{y}{80000} \qquad (4-7)$$

which means that U_H falls as H increases. Each voter has a most–preferred policy $H^*(y)$ at which the value of $u_H(H;y)$ defined by equation $(4-7)$ equals zero : her utility rises with H for $H < H^*(y)$ and falls with H for $H > H^*(y)$.

So each voter has single–peaked preferences. The median voter theorem applies : the level of highway construction chosen will be the median of the different voters' preferred levels $H^*(y)$. Squaring both sides of equation $(4-7)$,

$$\frac{9}{4}\frac{1}{H^*(y)} = [\frac{y}{60,000}]^2 \qquad (4-8)$$

or

$$H^*(y) = \frac{9}{4}[\frac{80,000}{y}]^2 \qquad (4-9)$$

which shows that $H^*(y)$ is a monotonically decreasing function of income (since here the publicly provided good has an income elasticity of demand of 0 : higher income people are no more willing to pay for highways than lower–income people, but they have to pay more taxes).

So the median voter is the voter of median income here : half the people — those of higher income — want less highway spending than the voter of median income, and the other half want more. Since the median income here is $80,000, equation $(4-9)$ becomes

$$H^*(y^{med}) = \frac{9}{4}[\frac{80,000}{60,000}]^2 = \frac{9}{4}\frac{16}{9} = 4$$

so that voters will choose to build 4 kilometres of highway is highway construction is determined by a direct vote, using pairwise majority rule.

*Q*5. If the head of some city's board of education wanted to make that city's education spending as large as possible, and if this head got to propose the city's education budget, how much spending would she propose if her budget had to be approved by a referendum of all the city's voters?

*A*5. The head of the board of education in this question has preferences similar to those of the budget–maximizing bureaucrats in Niskanen's theory of bureaucracy. She wants to spend more than the voters do, and if she has the exclusive power to propose the budget, she will be able to influence its size, even if it must be approved by voters.

So let Z represent the "quantity" of education provided by the city, some measure of the amount of resources students get. Assume that voters get benefits from this education : the benefits of voter i can be written $B_i(Z)$, with $B_i'(Z) > 0$, indicating that voters would prefer a better education system, at least if they did not have to pay for it. There is a cost to education. Denote that total cost by $C(Z)$, with $C'(Z) > 0$ since it will be more costly to provide better education. Let s_i be the share of the cost which must be paid by voter i (through his taxes). Then the net benefit from an education level to voter i will be

$$B_i(Z) - s_i C(Z) \tag{5-1}$$

Voter i's most–preferred level of education is the level which maximizes his net benefits. Differentiating expression $(5-1)$ with respect to Z, and setting it equal to zero, the preferred level of education Z_i^* of voter i is the level for which

$$B_i'(Z_i^*) - s_i C'(Z_i^*) = 0 \tag{5-2}$$

But voters cannot propose budgets for the board of education. They can only approve or reject proposals from the head of the board. So if the head of the board of education proposes to provide a level Z of education, at a total cost of T, then voter i will vote for the proposal if and only if he prefers that proposal to the situation if the budget is defeated.

If there will be no education spending at all (and no education) if the budget is defeated, then the voter will get a benefit $B_i(0)$, and will support the budget proposal if and only if

$$B_i(Z) - s_i T \geq B_i(0) \tag{5-3}$$

Now the budget T proposed by the head of the board of education must cover the actual costs of the proposed level Z of education : otherwise she could lose her job.

Moreover, if the head cares only about getting the largest possible budget passed, then she will want her proposed budget T to exactly cover the cost of the proposed level of education. If $T > C(Z)$, then the budget could be changed by increasing Z slightly, which would give the budget more support among voters. And if that change is possible, then the head of the board could increase Z a little more, and still have her budget approved, which is what she wants. So the head will always want to propose a budget in which

$$T = C(Z) \tag{5-4}$$

if she wants the largest possible level of education spending, and needs to get her budget passed.

Substituting from $(5-4)$ into $(5-3)$, voter i will vote for the budget if

$$B_i(Z) - s_i C(Z) \geq B_i(0) \tag{5-5}$$

Let $\hat{Z}_i$ be the **largest** value of Z for which equation $(5-5)$ holds. Note that

$$\hat{Z}_i > Z_i^* \tag{5-5}$$

(provided that $B_i'(0) > s_i C'(0)$) : Z_i^* maximizes the voters next benefit, and – as long as that net benefit is positive — there will be values of $Z > Z_i^*$ which the voter prefers to the alternative of $Z = 0, T = 0$.

So to get her budget approved, the head of the board must pick a level of education Z which is less than or equal to $\hat{Z}_i$ for at least half the voters. To maximize her budget, then, she should pick an education level Z equal to the median of the voters' $\hat{Z}_i$'s, and a budget equal tot he cost of that education level.

*Q*6. Suppose that the probability of a car accident was private information, which only each driver herself knew.

Is it possible for a government–run car insurance plan to provide the same level of coverage (in the event of an accident) to everyone, and to charge the same premium to everyone?

*A*6. The short answer is "yes". Problems in providing a "pooling contract", in which low–risk customers subsidize high–risk customers, stem from competitive private firms trying to attract a more profitable client base. If the government does not allow free entry of private competitors, then these problems will not arise.

For a little more detail, suppose that the probability of an accident claim for some person was π_i, which person i knows but which is not observable to anyone else. If the person is a risk averse von Neumann–Morgenstern expected utility maximizer, then her expected utility will be

$$EU_i = (1 - \pi_i)u(y - F) + \pi_i u(y - L + X - F)$$

from an insurance policy which requires her to pay F dollars in premiums, and which pays her X dollars in the event of an accident. Here y is her income, L is the loss from an accident, and $u(\cdot)$ is an increasing, concave, utility–of–wealth function. If I denote by $c_G \equiv y - F$ and $c_B \equiv y - L + X - F$ as consumption in the good and bad state, then the slope of the person's indifference curve through (c_G, c_B) is

$$s_i = -\frac{1 - \pi_i}{\pi_i} \frac{u'(c_G)}{u'(c_B)} \qquad (6-1)$$

Note that every customer who buys the insurance contract (F, X) will have the same c_G and c_B : it is only the probability of the two states which differs among people.

So if two people have the same insurance contract (F, X), then the person with the **lower** accident probability π_i will have an indifference curve through (c_G, c_B) which is more steep (when we graph c_G on the horizontal axis and c_B on the vertical).

Take any insurance contract (F, X) and its resulting consumption bundle (c_G, c_B). Since indifference curves of low–risk and high–risk people have different slopes, through the same point (c_G, c_B), that means that there always must be some other bundle (c_G', c_B') which (i) is preferred by lower–risk people to (c_G, c_B); (ii) is less attractive to higher–risk types than (c_G, c_B). This new consumption bundle must be below and to the right of (c_G, c_B), since the low–risk people have steeper indifference curves. A lower c'_B and a higher c'_G mean the new policy offers less coverage, but at a lower price.

So the previous three paragraphs indicate the following : if any insurance contract (F, X) is chosen by more than one risk class, and if that contract breaks even (or makes a profit), then there exists some other insurance contract (F', X') which will be preferred to (F, X) by lower–risk customers. If the original contract broke even with a mix of low– and high–risk customers, then the new contract (F', X') will make a profit serving only low–risk customers. But this "cream–skimming" by a competitor means that the original contract (F, X) attracts only higher–risk customers : if it broke even before the new contract (F', X') was introduced, now it will lose money. That is, in a competitive private insurance market, there will never be any contracts chosen in equilibrium by people in different risk classes.

But if there is only one firm allowed, and if it offers only one contract, that contract will be chosen by all the customers of different risk class, as long as people are sufficiently risk–averse that they all prefer the contract to going without insurance. That is, as long as

$$(1-\pi_L)u(y-F)+\pi_L u(y-L+X-F) > (1-\pi_L)u(y)+\pi_L u(y-L) \qquad (6-2)$$

for the lowest–risk customers, with accident probability π_L, then everyone, regardless of risk class, will choose to purchase the contract (F, X) if it's the only one on the market.

And the contract will break even, as long as

$$F \geq \bar{\pi} X \qquad (6-3)$$

where $\bar{\pi}$ is the average accident probability over the entire population. So the government does not need to know any individual's accident probability, just the average probability for the whole population.

Q7. How would a compulsory, fully–funded public pension programme affect overall saving in the following economy?

All people in the economy are identical. Each person earns \$2 million dollars over her working life. Each person's preferences can be represented by the utility function

$$U = C_y C_o$$

where C_y is total consumption over the person's working life, and C_o is total consumption when retired.

The person has no labour earnings when retired. Each dollar that she saves (during her working life) earns a return of 100%, and so yields her \$2 in consumption when retired.

The person can save as much as she wants from her earnings during her working life. But she cannot borrow.

The pension plan taxes the person $300,000 over her working life, and pays her a total pension income of $600,000 when she is retired.

*A*7. Suppose that a person chooses to save S dollars when working. Then her consumption when working will be

$$C_y = Y - T - S \tag{7-1}$$

where $Y = 2000000$ is her income in her working life, and $T = 300,000$ are the taxes she pays in her working life. If she saves S, her consumption when old will be

$$C_o = S(1+r) + P \tag{7-2}$$

where $P = 600,000$ is her pension income, and $r = 1$ is the interest she earns on her saving.

So she will choose an amount of saving to maximize $C_y C_o$, which, from equations $(7-1)$ and $(7-2)$, equals

$$U = C_y C_o = (Y - T - S)(S(1+r) + P) \tag{7-3}$$

Maximizing expression $(7-3)$ with respect to S yields a first–order condition

$$\frac{\partial U}{\partial S} = (1+r)(Y-T) - P - 2(1+r)S = 0 \tag{7-4}$$

or

$$S = \frac{Y-T}{2} - \frac{1}{2}\frac{P}{1+r} \tag{7-5}$$

Her savings decrease with the pension income she gets when old, and with the taxes she pays when young.

So the taxes of $300,000, and the pension income of $600,000, together reduce her savings by $300,000, compared with what she would save if there were no government pension plan. If there were no such plan ($T = P = 0$), she would choose to save 1 million dollars when working (since $Y = 2000000$). With the pension plan in place, she chooses to save $700,000.

The pension plan thus reduces private individual saving by $300,000. But the plan is fully funded. The government is collecting $300,000 in taxes from each working person, and investing it, at an interest rate of $r = 100\%$, to finance the pension of $600,000 it will pay. So the pension plan requires government saving of $300,000 per worker, which exactly offsets the reduction in private savings. Overall, the fully funded pension plan will have no effect on total saving here.

*Q*8. Does the current Canadian equalization programme (of grants from the federal government to provinces) equalize the cost of public output provision across provinces?

Explain briefly.

*A*8. An "ideal" equalization programme would equalize the cost of public output provision across provinces in the following sense : if province i wanted to provide the average level of provincial expenditure (averaged over all ten provinces, weighted by population), then the tax rate it would need to levy would be the average of the provinces' tax rates.

Suppose that there is only one tax base, and let b_i be the tax base per person in province i, with P_i the population of province i and t_i the tax rate levied by that province. Then if the average tax rate $\bar{t}$ and the average tax base $\bar{b}$ are defined as

$$\bar{b} = \frac{\sum_{j=1}^{10} b_j P_j}{\sum_{j=1}^{10} P_j} \qquad (8-1)$$

$$\bar{t} = \frac{\sum_{j=1}^{10} t_j b_j P_j}{\sum_{j=1}^{10} b_j P_j} \qquad (8-2)$$

the equalization payment to province i, in an ideal system, would be

$$e_i = (\bar{t})[\bar{b} - b_i] \qquad (8-3)$$

Now the average provincial tax revenue per person is

$$\bar{r} = \frac{\sum_{j=1}^{10} t_j b_j P_j}{\sum_{j=1}^{10} P_j} \qquad (8-4)$$

If province i happened to choose to levy the national average tax rate $\bar{t}$, then its tax revenue per person would be

$$r_i = \bar{t} b_i + e_i \qquad (8-5)$$

so that, from equation $(8-3)$,

$$r_i = \bar{t}\bar{b}$$

Definitions $(8-1)$, $(8-2)$ and $(8-4)$ imply that

$$\bar{r} = \bar{t}\bar{b} \qquad (8-6)$$

which proves the statement in the first paragraph.

So if the "cost of public output provision" in the question means the tax rate required to get a certain level of (nominal) public expenditure, then an ideal equalization system does equalize this cost across provinces, at least if the provinces chose to have the national average level of expenditure per person.

The actual Canadian system, however, differs from the ideal system in several key respects. First of all, it only averages over some of the provinces. Secondly, there are several revenue sources, not just one. And these revenue sources are not weighted equally in the formula. Most importantly, provinces collect equalization if their overall entitlement (the e_i of equation $(8-3)$ added up over all revenue sources) is positive. If it is negative, the provincial government does not need to pay in to the equalization programme. So, subject to the previous modifications, Canadian equalization equalizes the cost of public output provision across all provinces which are receiving equalization payments ; this "cost" will still be lower in provinces which are not entitled to receive equalization.