AP/ECON 4380 & GS/ECON 5950 : Answers to Midterm Exam

February 2016

Do all 3 questions. All questions count equally.

1. Suppose that there are 8 voters of type #1, 9 voters of type #2, 10 voters of type #3, 11 voters of type #4 and 12 voters of type #5, with the following preference orderings over 4 alternatives

	type #1	type #2	type #3	type #4	type #5
	[8 people]	[9 people]	[10 people]	[11 people]	[12 people]
first choice	y	y	x	x	w
second choice	w	z	z	w	z
third choice	z	x	y	z	y
fourth choice	x	w	w	y	x

Is any of the candidates a Condorcet winner under pairwise majority rule? Explain briefly.

answer : Note that to win a pairwise vote, a candidate must get the support of at least 3 of the 5 groups. [The smallest 3 groups, the people of types 1, 2 and 3, have 27 people altogether, and so will outvote the biggest 2 groups, those of types 4 and 5, which between them have 23 people.] So a candidate will be a Condorcet winner, if — and only if — she is preferred to any other candidate by at least 3 types of people.

Is w a Condorcet winner? Voters of type 1, 2 and 3 all prefer y to w, so that w loses a pairwise election against y, and therefore cannot be a Condorcet winner.

Is x a Condorcet winner? Voters of type 1, 2 and 5 all prefer y to x, so that x cannot be a Condorcet winner : it loses against y.

But y is not a Condorcet winner, since voters of types 3, 4 and 5 all prefer z to y.

The remaining possibility is that z is a Condorcet winner. However, voters of types 1, 4 and 5 all prefer w to z.

So, under pairwise majority rule, there is a cycle : w defeats z defeats y defeats x defeats w, and there is no Condorcet winner in this example.

2. Without providing details, sketch a proof of Arrow's Impossibility Theorem.

   answer One proof of the theorem, based on Arrow's own proof, involves the idea of "almost decisive groups". For some social ordering, some group G is almost decisive for alternative x over alternative y if : whenever every voter in G prefers x to y, and every voter outside of G prefers y to x, then the social ordering ranks x above y.

   One of the axioms of Arrow's Theorem, the Pareto principle, ensures that, for any alternatives x and y, there must exist a group G which is almost decisive, namely the group of **all voters**. The Pareto principle states that if all voters rank x above y, then x must be ranked above y in the social ordering.

   So there always exists some group which is almost decisive for x over y.

   Another step in the proof is the demonstration that : if a group G (of whatever size) is almost decisive for alternative x over alternative y, then for **any other** 2 alternatives v and w, that exact same group G must be almost decisive for v over w. This step depends on the assumptions that the social ordering obeys the Pareto principle, the principle of Univeral Domain, and the Independence of Irrelevant Alternatives.

   Now take any group G, which is almost decisive for x over y (and therefore for any other alternatives). Split that group into two parts, H and I, with each part containing at least one voter. We can always do this if the original group G has at least 2 members. If the social ordering obeys the three axioms P, UD, and IIA, then it must be true that either H or I is almost decisive.

   Starting with the group of all the voters, and repeating the previous step many times : since there always is an almost decisive group with all the voters, and since any almost decisive group has a strictly smaller subgroup which is also almost decisive [if the original group has at least 2 members], therefore **there must be an almost decisive group which consists of only one person**.

   The final step is to show that if a person is almost decisive over some pair of alternatives, that person must be a dictator. [A person is almost decisive for x over y if the social ordering ranks x above whenever that person ranks x above y, and everyone else ranks y above x. A person is a dictator for x over y if the social ordering ranks x over y whenever that person ranks x over y, whatever are the rankings of the other people.]

   (This is not the ony proof of Arrow's Theorem : sections 2.3 and 2.4 of Gaertner's text provide two completely different proofs, which are also fine.)

3. Suppose that all people in a jurisdiction had the same preferences, which could be represented by a utility function

$$u(c, g) = \sqrt{c} + \sqrt{g}$$

where c is private consumption expenditure, and g is government expenditure **per person**. People differ in their income y^i. The mean income $\bar{y}$ (in thousands of dollars per year) is 60, and the median income is 40.

In this jurisdiction, any public expenditure must be financed by a proportional income tax, so that the government's budget constraint is

$$\tau \bar{y} = g$$

where τ is the proportional income tax rate.

If two parties compete for votes by committing to provide some level of government expenditure g per person, what level will be chosen by the winning party? Explain briefly.

answer Since there are 2 parties, and they can commit to policies, then each party will choose the median of the voters' most–preferred policies in equilibrium, if voters have single–peaked preferences.

Voters do have single–peaked preferences here, since the indifference curves for the utility function $u(c, g) = \sqrt{c} + \sqrt{g}$ are convex to the origin. [The slope of an indifference curve, a curve with the equation $\sqrt{g} = u - \sqrt{x}$ is $dg/dx|_{u=\bar{u}} = -g/x$, which gets smaller in absolute value as we move down and to the right, if we graph x on the horizontal axis and g on the vertical.]

The preferences in this question are not quasi–linear, (as in the model in *Persson and Tabellini*), so to find who is the median voter, we have to find how voters' most–preferred levels of spending vary with their income.

Given that the average income in this jurisdiction is 60, the government budget constraint is

$$60\tau = g \qquad (3-1)$$

or

$$\tau = \frac{g}{60} \qquad (3-2)$$

A person's private consumption expenditure c is her after–tax income, $(1-\tau)y$ when she has income y and when the government levies a proportional income tax at a rate τ. From equation $(3-2)$, then

$$c = (1 - \frac{g}{60})y \qquad (3-3)$$

for a person of income y. That means that the person's utility will be

$$\sqrt{(1 - \frac{g}{60})y} + \sqrt{g} \qquad (3-4)$$

if the government provides per–capita spending g, and finances this spending by a proportional income tax. If expression $(3-4)$ is denoted as $W(g; y)$, then the derivative of this expression with respect to the level g chosen of public spending per capita is

$$\frac{1}{2}g^{-1/2} - \frac{1}{2}\frac{y}{60}[(1-\frac{g}{60})y]^{-1/2} \qquad (3-5)$$

The voter's most–preferred level of government spending, $g^*(y)$, is the value of g for which $W(g;y)$ is maximized, the value of g for which expression $(3-5)$ equals zero. So

$$\frac{1}{2}[g^*(y)]^{-1/2} - \frac{1}{2}\frac{y}{60}[(1-\frac{g^*(y)}{60})y]^{-1/2} = 0 \qquad (3-6)$$

or

$$\frac{1}{2}[g^*(y)]^{-1/2} = \frac{1}{2}\frac{y}{60}[(1-\frac{g^*(y)}{60})y]^{-1/2} = 0 \qquad (3-7)$$

or (squaring both sides of $(3-7)$)

$$g^*(y) = \frac{3600}{y}(1-\frac{g^*(y)}{60}) \qquad (3-8)$$

so that

$$g^*(y) = \frac{3600}{y+60} \qquad (3-9)$$

The most–preferred level of spending of the voter of income y, defined by equation $(3-9)$, is a decreasing function of the person's income y. [That's because, with these preferences, a person's income elasticity of demand for the publicly provided good is less than her price elasticity of demand for the publicly provided good.]

So the higher the person's income y, the lower is her preferred public expenditure (per person) level $g^*(y)$. That means that the median voter is the voter of median income : everyone with lower income wants more government spending, and everyone with higher income wants less government spending.

Therefore, each political party will choose the same platform, $g^*(40)$, if the median income of the voters is 40. From equation $(3-9)$, that means a level of public expenditure per person of 36.