answers to some of the sample exercises : Public Choice

Ques 1 The following table lists the way that 5 different voters rank five different alternatives.
Is there a Condorcet winner under pairwise majority rule for this example?
Explain briefly.

	voter #1	voter #2	voter #3	voter #4	voter #5
first choice	w	v	x	y	z
second choice	v	x	z	z	w
third choice	x	y	v	v	v
fourth choice	y	z	w	w	y
fifth choice	z	w	y	x	x

answer No, there is no Condorcet winner here. No real way of checking this other than looking at the pairwise matches, and seeing whether one alternative can defeat all of the others.

Alternative v looks like a reasonable candidate for a Condorcet winner, since most of the voters rank it fairly highly.

How does it fare in one–on–one elections? It defeats the alternative w by a vote of 3 to 2 : voters #2, #3 and #4 all rank v above w. [That means that w can't be a Condorcet winner, since it loses to v.]

Alternative v defeats x by a vote of 4 to1 : every voter except #3 ranks v above x.

Alternative v also defeats y by a vote of 4 to 1, since every voter except #4 prefers v to y.

That leaves alternative z : if v defeats z as well, then v is a Condorcet winner ; if not, then z is the only possible Condorcet winner.

Alternative z does defeat v, by a vote of 3 to 2. Voters #3, #4 and #5 all rank z above v.

So either z is a Condorcet winner, or there is no winner at all [since v loses to z, which means that v isn't a Condorcet winner, and w, x and y all lose to v, which means that none of those three alternatives can be a Condorcet winner].

Against the alternative y however, z loses : voters #1, #2 and #4 all rank y above z.

So for each of the five alternatives, there is some other alternative which is ranked higher by a majority of the voters.

Ques 3 Give an example in which there is some arrangement of the alternatives such that (i) preferences are single–peaked for every voter except one ; and (ii) there is no Condorcet winner.

answer A simple example is the "standard" three–person, three–alternative example, in which voter #1 prefers x to y to z, voter #2 prefers y to z to x and voter #3 prefers z to x to y. In this example, if we arrange the alternatives with x on the left, y in the middle and z on the right, then voters #1 and #2 both have single–peaked preferences, but voter #3 does not. [Or we could put y on the left, z in the centre and x on the right, and then, voters #2 and #3 would have single–peaked preferences but voter #1 would not.]

And there is no Condorcet winner in this example, under majority rule, since x defeats y by a vote of 2 to 1, y defeats z by a vote of 2 to 1, and z defeats x by a vote of 2 to 1.

Ques 5 Suppose that policies could be represented by pairs of numbers (such as the sales tax rate, and the level of carbon emissions allowed). Suppose that each of 5 voters i had a preferred policy (x_i, y_i), and ranked policies by how far they were from her preferred policy (the further away from her preferred policy, the less she likes it).

If the preferred policies of the 5 voters were $(1,1)$, $(1,5)$, $(3,3)$, $(5,1)$ and $(5,5)$, would there be a policy which defeats all others in a pairwise vote? If so, which one? If not, why not?

answer This is an example of a "median in all directions", so that there does exist a Condorcet winner, the policy $(3,3)$.

Graphically, any line through the point $(3,3)$ has 2 of the other four voters' preferred policies on one side of the line, and 2 of the other four voters' preferred points on the other side.

So take any other policy (x,y), and run it against $(3,3)$. Who will vote for (x,y)? Anyone whose preferred policy is closer to (x,y) than it is to $(3,3)$. To find which preferred policies are closer to (x,y), take the line segment connecting (x,y) to $(3,3)$. Now take the point exactly halfway along this segment, equidistant from $(3,3)$ and (x,y). Finally, take the line though this halfway point, which is perpendicular to the original line segment. Every point on the same side of this new line segment as $(3,3)$ is, will be closer to $(3,3)$ to (x,y). And because $(3,3)$ is a "median in all directions", at least 2 of the other 4 voters will have a preferred point on the same side of this line as $(3,3)$.

Ques 6 "If, for a given set of voters and alternatives, there exists a Condorcet winner, then the Condorcet winner will get the highest score using the Borda count." True or false? Explain briefly.

answer The statement is false. A Condorcet winner must get at least the average Borda count, but will not necessarily get the highest.

For example, suppose that we had 3 alternatives, x, y and z, and 19 voters. 10 of the voters prefer x to y to z, and the other 9 prefer y to z to x. Alternative x is a Condorcet winner : it defeats y by a vote of 10 to 9, and z by a vote of 10 to 9. But the Borda counts (with 2 for a first choice, 1 for a second choice and 0 for a third choice) are 20 for x, 28 for y and 9 for z.

Ques 7 Give an example of a voting rule in which adding a "clone" (a new alternative which is very similar to one of the existing alternatives) could change the outcome.

answer A very simple example of such a voting rule is plurality rule. Suppose that 60 percent of the voters prefer x to y. Now suppose that alternative z is very similar to x, so that half of the x supporters rank the alternatives x preferred to z preferred to y and the other half of them rank them z preferred to x preferred to y, whereas all of the y supporters still prefer y to either x or z. Then y would win under the plurality rule, with 40 percent of the vote (versus 30 percent for each of z and x).

As another example of such a rule, consider the Borda count. Take the example in the previous question 6. Now add a fourth alternative x', which everyone agrees is very similar to x, but slightly inferior to x. Then 10 of the voters will have the ranking x preferred to x' preferred to y preferred to z, and the other 9 will have the ranking y preferred to z preferred to x preferred to x'. The new Borda count scores will be x gets 39, x' gets 20, y gets 37 and z gets 18. So adding the clone x' makes x the Borda count winner, whereas y had been the winner before.

Ques 8 If preferences over candidates for some elected office happened to be single–peaked, would the median of the most–preferred candidates win under plurality rule? Explain briefly.

answer Not necessarily true. The Condorcet winner does **not** have to win under plurality rule.

Suppose we arrange the candidates with L on the left, C in the centre and R on the right. Suppose that 5 voters have the ranking L preferred to C preferred to R, 2 of the voters have the ranking C preferred to L preferred to R, and 6 of the voters have the ranking R preferred to C preferred to L. Here everyone's rankings are single–peaked. The candidate C is the median of the most–preferred candidates [since 8 of the 13 voters have a preferred policy of C or something to the right of C, and 7 of the 13 voters have a preferred policy of C or something to the left of C].

But under plurality rule, R wins, with the most first–place votes.

Ques 9 Suppose that there are 11 voters of type 1, 12 voters of type 2, 13 voters of type 3, 14 voters of type 4 and 15 voters of type 5, with the following preference orderings over candidates :

	[11 people]	[12 people]	[13 people]	[14 people]	[15 people]
first choice	v	w	x	y	z
second choice	w	x	v	w	v
third choice	x	y	w	v	x
fourth choice	y	z	y	z	w
fifth choice	z	v	z	x	y

Describe the outcome of the following voting rules, for the population of voters described in the table :

1. Plurality rule (the candidate with the most first–place votes wins).
2. Plurality with a single run-off (if no candidate has a majority of the first–place votes, a second run–off election is conducted between the two candidates with the most first–place votes).
3. "Hare system" : if no candidate has a majority of first–place votes, the candidate with the least first–place votes is eliminated, and the procedure repeated until there is a majority winner.
4. "Coombs system" : if no candidate has a majority of first–place votes, the candidate with the most last–place votes is eliminated, and the procedure repeated until there is a majority winner.
5. Borda count.
6. "Nanson system" : do a Borda count, eliminate all candidates with Borda scores less than the average, and repeat until there is one candidate left
7. "Baldwin system" : do a Borda count, eliminate the candidate with the lowest Borda score ; redo this process 3 more times until only one candidate is left
8. "Black's system" : choose a Condorcet winner if there is one ; otherwise choose the candidate with the highest Borda score

answer The winner under many of these rules can be checked using the website tool from the extended readings for section 2. The output from that website is here.

It can be checked that the alternative v is a Condorcet winner : it defeats w by a vote of 39 to 26, it defeats x by a vote of 40 to 25, it defeats y by a vote of 39 to 26, and it defeats z by a vote of 38 to 27.

So v must win under Black's system, or under the Nanson system, since those rules are "Condorcet efficient" (they always select the Condorcet winner if there is one).

plurality : Under plurality, the alternative z gets the most first place votes (15). If there were plurality with a run–off, there would be a run–off between z and y, since those two alternatives are ranked first by the most voters (and neither of them is ranked first by a majority of voters). In the run–off between y and z, y wins by a vote of 50 to 15, since only type–5 voters prefer z to y.

Hare : Under the Hare system, v is eliminated first, since it gets the fewest first–place votes (11). All 11 of v's votes get transferred to the next–most–preferred alternative of type–1 voters, w. So in the second round, no alternative gets a majority (w leads, with 23 out of 65 votes), and the alternative with the fewest first–place votes among these remaining 4 alternatives gets knocked off the ballot. That's alternative x, with 13 votes. All of the votes of the type–3 voters, who preferred x now get transferred to their best choice remaining, which is alternative w. (Alternative v has already been knocked off the ballot.) So on the third ballot, alternative w gets 36 votes (all the votes of type–1 voters, type–2 voters and type–3 voters. That's a majority of the 65 voters, so that w is the winner under the Hare system.

Coombs : Under the Coombs system, the alternative with the most last–place votes gets knocked off the ballot, if no alternative has a majority of the first–place votes. That's z, with 24 last–place votes. Once z has been knocked off the ballot, the first–place votes of the type–5 voters, for whom z was their favoured alternative, get allocated to their next–most–preferred alternative, v. On the second ballot, no alternative has a majority of the first–place votes (v leads with 26 votes), so the alternative with the most last–place votes of the 4 remaining alternatives gets eliminated. That's y which is now ranked last by 39 voters (since it is now last for type–1 and type–3 voters, now that z is off the ballot).

On the third ballot, with the three choices v, w and x remaining, the first–place votes of type–4 voters, who ranked the now–knocked–off alternative y first, get allocated to their second–place choice, which is alternative w. So on the third ballot, v gets 26 votes (from types 1 and 5), w gets 26 as well (from types 2 and 4), and x gets the remaining 13 votes. Still no candidate has a majority. We have to go to a fourth ballot.

Which of the remaining alternatives v, w and x gets knocked off for the 4th ballot? Under the Coombs system, it's the one which is ranked third among these remaining three by the most voters. That's w, since the 28 voters of types 3 and 5 all rank it below v and x, whereas the 25 people of types 1 and 4 rank x last, and the 12 people of type 2 rank v last.

So on the fourth ballot, it's v versus x. And v wins that election, since it is a Condorcet winner, and thus defeats any other alternative in a 2–way election.

Borda : If we give 2 points for a first–place choice, 1 for a second–place choice, 0 for a third–place choice, -1 for a fourth–place choice and -2 for a last–place choice, the Borda count scores are : v:+26, w: +34, x: +10, y : -26 and z :-44. So w wins.

Baldwin and Nanson : In this example, alternative v is a Condorcet winner. And the Baldwin and Nanson procedures are "Condorcet efficient" : if there is a Condorcet winner, these rules always choose it. The reason : a Condorcet winner must always get at least the average score in a Borda count ; so a Condorcet winner will never get knocked out at any stage of the Nanson or Baldwin procedures ; eventually these procedures will lead to a one–on–one contest between the Condorcet winner and some other alternative ; and by definition a Condorcet winner must win such a one–on–one election. So here the Nanson and Baldwin procedures select alternative v.

Ques 10 Suppose that the social choice rule ranks y above x when the two voters' preferences are as listed in the table below. Suppose as well that the social choice rule is transitive, and obeys the Pareto principle ("unanimity") and the Independence of Irrelevant Alternatives.

Show that the social choice rule must always rank y above z whenever voter #2 ranks y above z.

	person #1	person #2
first choice	x	y
second choice	z	x
third choice	y	z

answer The question stated that the social choice rule ranked y above x (with the preferences as in the table in the question). Both voters rank alternative x higher than alternative z. So the Pareto principle says that the choice rule must rank x above z. Now transitivity of the overall social choice rule says that : if it ranks y above x, and if it also ranks x above z, then it must rank y above z.

Now consider changing the preferences of either voter. If we change the preferences of voter #1, but leave her ranking z above y as she did originally, then the axiom of the Independence of Irrelevant Alternatives says that the social choice rule cannot change its ranking of y against z, since neither voter has changed the way he or she ranked those two alternatives.

On the other hand if we change the preferences of voter #1, so that she now prefers y to z, then the Pareto principle says that the social choice rule must rank y above z.

Finally consider changing the rankings of voter #2. As long as he still ranks y above z, then the IIA axiom says that the social choice rule cannot change its ranking of y against z, since neither voter has changed her or his ranking of these two alternatives.

So for all the 18 possible profiles, for which voter #2 prefers y to z, the social choice rule must rank y above z.

Ques 11 Which of the axioms of Arrow's Impossibility Theorem does the following social choice rule violate? The rule : if person #1 and person #2 have exactly the same rankings of the alternatives, then the social choice rule will exactly coincide with their ranking ; if their rankings are different in any way, then the social choice rule will coincide with person #3's ranking.

answer If everyone ranks alternative x above alternative y, then in particular voters #1 and #3 must both rank alternative x above alternative y. The social choice rule uses either the ranking of person #1 **or** the ranking of person #3 (depending on whether person #2 exactly agrees with person #1). So the rule obeys the Pareto principle : whenever everyone ranks alternative x above alternative y, then so does the rule.

Since the preference orderings of person #1 and of person #3 are both transitive, then the social choice rule always generates a transitive ordering.

Person #1 is not a dictator : the social choice rule may go against her preferences if she disagrees with person #2. And no–one else is a dictator : the rule may go against person #3's preferences if person #1 and person #2 happen to agree exactly. And it may go against person #2 if he disagrees with both person #1 and person #3.

The rule will work whatever is the profile of preferences, however many voters and alternatives there are. So it satisfies the property of universal domain.

Therefore, Arrow's Impossibility Theorem says that the rule must violate the axiom of the Independence of Irrelevant Alternatives, since it satisfies the other 4 axioms.

More directly, suppose that person #1 and person #2 both prefer x to y to z,and person #3 prefers z to y to x. Then the rule ranks x above y above z, since person #2 agrees completely with person #1. Now change person #2's ranking to : person #2 prefers y to x to z, leaving unchanged the preferences of person #1 and of person #3. Since person #2 now does not agree completely with person #1, then the rule says that we use person #3's rankings : the social ordering is z above y above x. So before, the social choice rule ranked x above z ; after the change in person #2's rankings the social choice rule ranks z above x. Thus the social choice rule has changed the way it ranked x versus z. But **no person** has changed the way she or he ranks these two alternatives : person #1 and person #3 have not changed their rankings at all, and person #2 ranked x above z both before and after the change in his rankings.