Two Types of Dictator

Person *i* is a “Vickrey dictator for x over y ” if, whenever *i* prefers x to y and every other person feels **exactly the opposite** way, preferring y to x , then the social ordering ranks x above y .

Person *i* is an “Arrow dictator for x over y ” if, whenever *i* prefers x to y , then the social ordering ranks x above y (whatever anyone else’s rankings are).

PROPOSITION V1 : If person is a Vickrey dictator for x over y , then she must also be a Vickrey dictator for x over **any other** alternative z — if the social choice rule obeys *P* and *IIA*.

Proof of Proposition V1

Suppose that person 1 is a Vickrey dictator for x over y .

	voter #1	everyone else
first choice	x	y
second choice	y	z
third choice	z	x

Since person #1 is a Vickrey dictator for x over y , the social ordering must rank x above y .
P : the social ordering must rank y above z .
Transitivity : the social ordering must rank x above z for the above profile

even though everyone except for person 1, ranks z above x .

IIA : the social ranking of x over z cannot change when the profile of preferences changes, as long as no voter changes her ranking of x versus z .
So whenever person 1 prefers x to z , and the rest of the voters prefer z to x , then the social ordering must have x above z , which means that person 1 is a Vickrey dictator for x over z . •

PROPOSITION VA : If a person is a Vickrey dictator for x over y , then she is also an Arrow dictator for x over y (and vice versa).

	voter #1	group A	group B
first choice	x	z	z
second choice	z	x	y
third choice	y	y	x

Since person #1 is a Vickrey dictator for x over y , Proposition V1 shows that she is also a Vickrey dictator for x over z . So the social ordering must rank x above z , since person #1 ranks x above z , and everyone else ranks them the opposite way.
P : the social ordering must also rank z above y
transivity : the social ordering must rank x above y for this profile of preferences

IIA : the social ordering must rank x and y the same way, whenever person #1, group *A*, and group *B* rank x and y the way they do in the diagram above

so the social ordering **must** rank x above y , whenever person #1 ranks x above y , and people in group *A* rank x above y , and people in group *B* rank y above x

so person #1 is an Arrow dictator

Almost Decisive Groups

Definition : A group *G* of voters is “almost decisive” for x against y if the social ordering ranks x above y , whenever every person in group *G* ranks x above y , **and** when every person outside of group *G* feels the opposite way (ranking y above x).

Note : If the group *G* has just 1 person, then that person being almost decisive is the same thing as that person being a Vickrey dictator.

Arrow–Vickrey–Sen Proof

Take any pair of alternatives, x and y . Now let *G* be a group of voters which is almost decisive for x over y .
How do I know that there is such a group?

Profile 1

	G	*NG*
first choice	v	y
second choice	x	w
third choice	y	v
fourth choice	w	x

The social ordering for profile 1 must rank v above w .

Why?

Since G is almost decisive for x versus y (and since everyone in NG ranks y above x here), the social ordering must rank x above y .

P : the social ordering must rank y above w

P : the social ordering must rank v above x

transitivity : the social ordering must rank v above x , x above y , and y above w .

Modifying Profile 1

change the profile, but make sure that every person in *G* still ranks v above w , and that every person in *NG* ranks w above v. *IIA* : the social ordering ranks w above v, since we have not changed the way any person ranks v versus w , and the social ordering ranked v above w in Profile 1.

So : if group *G* is almost decisive for x over y , then it's almost decisive for v over w .
i.e. if a group is decisive for **one** pair of alternatives, that same group must be decisive for **any other** pair of alternatives.

Splitting the Group

Suppose that group G has 2 or more people in it.
Split it in 2, into group A and group B. Make sure that all the people in G are in either A or B, and that the memberships of the two groups do not overlap.

Profile 4

	A	B	NG
first choice	x	y	z
second choice	y	z	x
third choice	z	x	y

2 cases possible

Case 1 : social ordering ranks x above y

Everyone in *A* and in *B* ranks y above z . And group *G* (which consists of *A* and *B* together) is almost decisive for y versus z (or any other pair of alternatives). So the social ordering must rank y above z .
transitivity : the social ordering must rank x above z
everyone in *A* ranks x above z , and everyone else — those in *B* and *NG* — rank z above x
So in this case, *A* is **almost decisive** for x versus z .

Case 2 : social ordering ranks y above x

Everyone in group *B* ranks y above x .
Everyone in groups *A* and *NG* ranks x above y .
That means that group *B* is almost decisive for y versus x .

Where does it Stop?

We started with a group G which was almost decisive for any pair of alternatives.
This group was now split into two smaller pieces.
One of these two smaller groups must also be almost decisive for any pair of alternatives.
We can now take this new, smaller decisive group, and split it in 2. So we can keep going, finding ever–smaller almost–decisive groups — until we can't split any further.
That happens when our almost–decisive group has been whittled down to a single person.

Punchline

if the social ordering obeys *UD*, *P* and *IIA*, almost decisive groups can be split again and again

until we have an almost decisive group containing 1 person

which is the same thing as a Vickrey dictator

which is the same thing as an Arrow dictator