Single Crossing

1 An Example

This example is slightly different than Example 1 (pg. 24) in *Persson and Tabellini*. The main feature, however, is that voters differ in a single dimension (here, their ability), and this dimension determines voters' policy preferences.

Here voters choose a single policy, the tax rate q — just as in example 1 in *Persson and Tabellini*. As in that example, each voter i cares about her consumption level c^i, and how much leisure she has x^i. In particular, each voter's utility function is

$$c^i + V(x^i) \tag{1}$$

where $V(x^i)$ is an increasing, strictly concave function. So expression (1) says that the person wants to consume more, and to work less. But it also implies a very particular relation between leisure and utility : the marginal utility of leisure $V'(x^i)$ does not depend on the person's consumption. In other words, she has **quasi–linear** preferences.

Here's the difference between this version of the example, and that in *Persson and Tabellini* : here each person has the same endowment of time. So if the fraction of time which person i chooses to work is l^i (as in *Persson and Tabellini*), then

$$l^i + x^i = 1 \tag{2}$$

(So, in comparison with equation (2.5) in *Persson and Tabellini*, everyone has the same α^i.)

Each person's productivity is different. Let ω^i be person i's productivity. (In *Persson and Tabellini*, people all have the same productivity, $\omega^i = 1$.)

Each person is taxed at the same tax rate q, and each person receives the same grant f from the government. So person i's total consumption, which equals her after–tax wage income, plus her grant income, is

$$c^i = (1-q)\omega^i l^i + f \tag{3}$$

2 Policies

The policy which people choose, using pairwise majority rule, is the tax rate q^i.

What happens to the money collected from this tax?

As in the example in *Persson and Tabellini*, the money is given back to each person (equally), in the form of a grant. So the size of the grant f is determined by the amount of tax revenue collected. The average tax revenue collected per person is the tax rate q, times the average wage income of people, so that

$$f = q[\bar{\omega^i l^i}] \tag{4}$$

where $\bar{\omega^i l^i}$ is the average income : the total income of all the voters, divided by the number of voters.

Now the labour supply l^i of each person will be chosen be that person, and it will depend on what the tax rate is. We'll get there (in the next section). But what is important here is that the grant f depends on the tax rate q. I'll write $f(q)$ to show that the grant income does depend on the tax rate q.[1]

People vote over tax rates q, taking into account the effect the tax rate has on their grant income (equation (4), as well as the effect the tax rate has on their own after–tax wage income.

3 Labour Supply

If the tax rate is q, and if the person receives grant income $f(q)$, then her utility will be $c^i + V(1 - l^i)$, which, from equation (3) is

$$f(q) + (1-q)\omega^i l^i + V(1 - l^i) \tag{5}$$

The person chooses her labour supply l^i so as to maximize her utility (5). Choosing l^i so as to maximize this expression means setting the derivative of this expression with respect to l^i equal to 0, or

$$(1-q)\omega^i - V'(1 - l^i) = 0 \tag{6}$$

The assumption that the function $V(x^i)$ is concave implies that :

PROPOSITION 1: A person's labour supply l^i is an increasing function of her wage rate ω^i.

PROOF : If $\omega^k > \omega^i$, then $(1-q)\omega^k > (1-q)\omega^k$. Equation (6) then implies that $V'(1 - l^k) > V'(1 - l^i)$. Since $V(x)$ is concave, $V'(x)$ derceases with x, so that $V'(1 - l^k) > V'(1 - l^i)$ means $1 - l^k < 1 - l^i$, or $l^k > l^i$. •

Note : the assumption that preferences are quasi–linear is necessary here : it eliminates the income effect of wages on labour supply, and guarantees that a person's labour supply is an increasing function of her wage rate.

[1] From equation (4), the tax rate q has 2 effects on average tax revenue per person. First of all, this revenue is directly proportional to the tax rate, so that increases in q lead to more tax revenue. But people's choice of labour supply, l^i will depend on the tax rate, and it turns out here that l^i must fall when the tax rate rises. So the overall impact of q on f could go either way.

4 Utility from a Policy

Write a person's labour supply as $l^i(q, \omega^i)$, since it depends on both the tax rate and on her own (gross–of–tax) wage. Then her utility, if the tax rate q is chosen, and if she choose a labour supply of l^i, will be

$$f(q) + (1-q)\omega^i l^i + V(1-l^i)$$

How does this utility vary with the tax rate? The derivative of this expression with respect to q is

$$\frac{dU^i}{dq} = f'(q) - \omega^i l^i + \frac{\partial l^i}{\partial q}[(1-q)\omega^i - V'(1-l^i)] \tag{7}$$

The last terms in expression (7) are there because a change in the tax rate will cause the person to change her own labour supply.

But equation (6) shows that the term in square brackets must equal zero.[2] So

$$\frac{dU^i}{dq} = f'(q) - \omega^i l^i \tag{8}$$

which has the following implication :

PROPOSITION 2: Take a particular tax rate q. Then $\frac{dU^i}{dq} > \frac{dU^k}{dq}$ if and only if $\omega^i < \omega^k$.

PROOF : Proposition 1 shows that a person's labour supply increases with her wage rate. So $\omega^i l^i > \omega^k l^k$ if and only if $\omega^i > \omega^k$, and then equation (8) shows that $\frac{dU^i}{dq}$ is a decreasing function of ω^i. •

5 Single Crossing

Take two different tax rates, q'' and q', with $q'' > q'$. Suppose that a person of ability ω^i prefers q'' to q'. That will be true if and only if

$$U^i(q'') - U^i(q') = \int_{q'}^{q''} \frac{dU^i}{dq} dq > 0 \tag{9}$$

Proposition 2 says that $\frac{dU^k}{dq} > \frac{dU^i}{dq}$ whenever $\omega^k < \omega^i$. So if a person of ability ω^i prefers tax rate $q'' > q'$ to tax rate q', then so will a person of ability $\omega^k < \omega^i$.

PROPOSITION 3 : If $q'' > q'$ and $\omega^i > \omega^k$, and if person i prefers policy q'' to policy q', then person k must also prefer the higher–tax policy q'' to q'.

This is exactly the "single–crossing" property defined in Definition 3 (pg. 23) of *Persson and Tabellini.*

[2] This is an application of the Envelope Theorem, for those of you who have seen this theorem in microeconomic theory.

What does that mean? It means that, in this example, there must be a Condorcet winner. The tax policy q which will be chosen, under pairwise majority rule, is the tax rate most preferred by the person with the **median** ability level. That is, the median ability level ω^m is the ability level such that half the people have higher ability (than ω^m) and half the people have lower ability. NOTE : This median level of ability will **not**, in general, be equal to the average (or "mean") ability level.

Why is this policy a Condorcet winner? Take any tax rate $q' < q$. In a pairwise vote between q and q', Proposition 3 says that every person of ability level ω^m or less will prefer q to q' : that's a majority of voters, so q defeats q'. Now take a tax rate $q'' > q$. Proposiiton 3 says that any voter whose ability is greater than the median will prefer q to q'' : again, that's a majority of voters. So q will defeat any alternative, higher or lower, in a pairwise vote, if q is the most–preferred tax rate of the person of median ability.