GS/ECON 5010 Answers to Assignment 2 October 2004

1. Added to the original table of prices and quantities at each of the four times are the costs c^{ij} of the four preferred bundles in the four years. So, for example, the entry under c^{3j} in the first row is the cost of the bundle x^3, in year 1 prices. (So $c^{ij} = \mathbf{p}^j \cdot \mathbf{x}^i$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t	c^{1j}	c^{2j}	c^{3j}	c^{4j}
1	1	1	5	6	10	4	36	32	40	64
2	5	1	2	2	15	3	48	31	45	50
3	2	5	2	5	10	5	70	85	70	78
4	5	4	1	4	10	10	74	73	70	70

If $c^{ij} \leq c^{jj}$ (for $i \neq j$), then bundle $\mathbf{x}^j$ has been revealed preferred to bundle $\mathbf{x}^i$: the consumer could have afforded $\mathbf{x}^i$ but instead chose $\mathbf{x}^j$.

So the first row shows that $\mathbf{x}^1$ is revealed preferred to $\mathbf{x}^2$; the third row shows that $\mathbf{x}^3$ is revealed preferred to $\mathbf{x}^1$; the fourth row shows that $\mathbf{x}^4$ is revealed preferred to $\mathbf{x}^3$. The three entries c^{21}, c^{13}, and c^{34} are the only entries for which the cost of some other $\mathbf{x}^i$ is less than or equal to the cost of the bundle $\mathbf{x}^j$ actually chosen in year j.

That means that the preferences are consistent with $WARP$, and with $SARP$: there are no cycles. The data are consistent with a person for whom $\mathbf{x}^4 \succ \mathbf{x}^3 \succ \mathbf{x}^1 \succ \mathbf{x}^2$ (and consistent only with that preference ordering).

2. If the person pays B for a bet, and expects to win P with probability 0.5, then her expected utility is

$$(0.5)U(W-B) + (0.5)U(W-B+P)$$

If the person is indifferent between taking this bet, and not, then it must be true that

$$(0.5)U(W-B) + (0.5)U(W-B+P) = U(W)$$

With the specified utility function, that means that

$$(W-B)^\alpha + (W-B+P)^\alpha = 2W^\alpha$$

implying that

$$W - B + P = [2W^\alpha - (W-B)^\alpha]^{1/\alpha}$$

so that

$$P = [2W^\alpha - (W-B)^\alpha]^{1/\alpha} + B - W \qquad (2-1)$$

defines the prize P as a function of the amount bet B, and of the person's wealth W.

If equation $(2-1)$ is differentiated with respect to B,

$$\frac{\partial P}{\partial B} = [2W^\alpha - (W-B)^\alpha]^{1/\alpha - 1}(W-B)^{\alpha-1} + 1 \qquad (2-2)$$

Substituting from equation $(2-1)$,

$$\frac{\partial P}{\partial B} = \big[1 + \frac{P}{W-B}\big]^{1-\alpha} + 1 \qquad (2-3)$$

Equation $(2-3)$ shows that $\partial P/\partial B = 2$ at $B = 0$. If the bet is very small, the person is willing to accept a bet at almost actuarially even odds ($P = 2B$). Equation $(2-3)$ also shows that P is a **convex** function of B ; as the bet increases in magnitude, the person needs a larger risk premium to take the bet.

Equation $(2-1)$ can also be written

$$\frac{P}{W} = \big[2 - (1 - \frac{B}{W})^{\alpha}\big]^{1/\alpha} + \frac{B}{W} - 1 \qquad (2-4)$$

so that, with these preferences, the ratio of the required prize to the person's wealth depends only on the fraction of her wealth which she must bet. The diagram graphs equation $(2-4)$ (for the case $\alpha = -0.5$). Because the curve is convex, and because it starts at the point $(0,0)$, the ratio P/B must increase as we move up the curve. Also, the slope of the curve must everywhere be steeper than the slope of a line connecting the curve with the origin.

[Why? By definition, if $f(x)$ is a strictly convex function, then

$$\frac{f(x) - f(x_0)}{x - x_0} < f'(x)$$

for any $x > x_0$. Letting $x_0 = 0$ then implies that $f'(x) > f(x)/x$, so the slope of the curve at any point is greater than the slope of a line connecting the curve with the origin. How does the slope of that line connecting the curve with the origin vary? If

$$g(x) \equiv \frac{f(x)}{x}$$

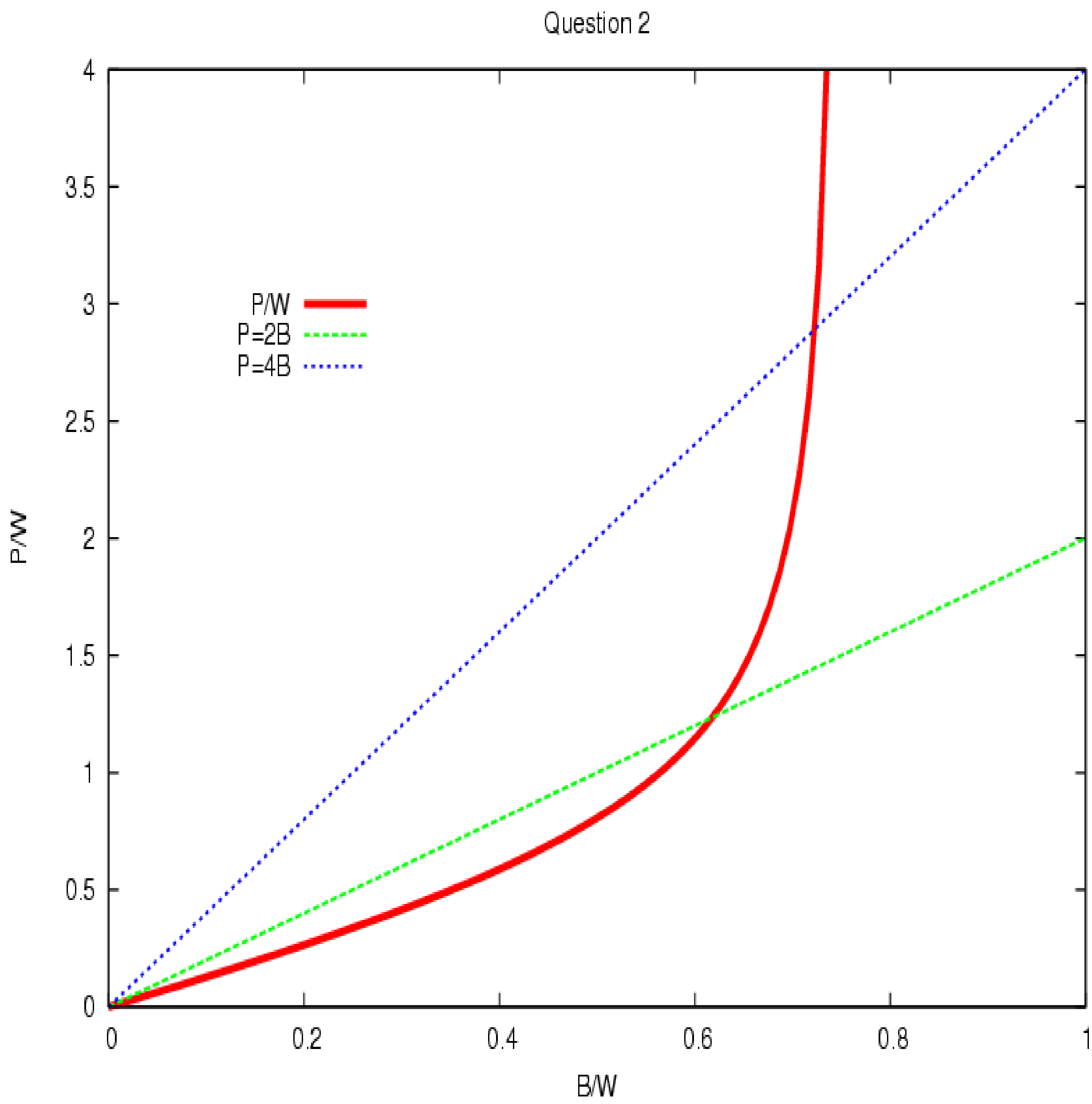
then

$$g'(x) = \frac{1}{x}(f'(x) - \frac{f(x)}{x})$$

which must be positive if $f(\cdot)$ is convex, and if $f(0) = 0$.]

These properties of the graph can be used to show that, for a given bet B, the required prize P is a decreasing function of wealth W. The particular property needed is that P/B increases as B/W increases. (That is, the slope of the line connecting the origin with the curve increases as we move up the curve.) Suppose that $W' > W$, and consider the prize required to get a person to accept a bet of a given amount B. If we write this prize as $P(B, W)$, then equation $(2-4)$ says that

$$P(B, W') = \frac{W'}{W} P(\frac{W}{W'} B, W) \qquad (2-5)$$



Question 2
P/W
P=2B
P=4B
P/W
B/W
0
0.5
1
1.5
2
2.5
3
3.5
4
0
0.2
0.4
0.6
0.8
1

Since P/B increases with B (for given wealth), then

$$P(\frac{W}{W'}B, W) < \frac{W}{W'}P(B, W) \qquad (2-6)$$

when $W' > W$. Plugging inequality $(2-6)$ into $(2-5)$ shows that $P(B, W)$ must be a **decreasing** function of W. This is, perhaps, not surprising, since the given preferences exhibit decreasing absolute risk aversion, so that wealthier people require a smaller risk premium.

3. Let x represent the return on the gamble. This is a random variable. It can have any distribution function — as long as the maximum possible return X is small enough that the person's marginal utility of wealth $\alpha - 2\beta(W + X) > 0$.

If the person's initial wealth is W, and if she is just willing to take the gamble, then it must be the case that

$$\alpha W - \beta W^2 = \alpha E(W + x) - \beta E[W + x]^2 \qquad (3-1)$$

The left side of $(3 - 1)$ is her utility if she does not take the gamble, and the right side is her expected utility if she does take the gamble. Using the facts that $E(x + y) = E(x) + E(y)$ for any random variables x and y, and that $E(ax) = aE(x)$ for any non–stochastic a and random x, equation $(3 - 1)$ becomes

$$\alpha W - \beta W^2 = \alpha W + \alpha E(x) - \beta W^2 - 2\beta W E(x) - \beta E(x^2) \qquad (3-2)$$

which can be simplified to

$$\alpha E(x) = \beta[2W E(x) + E(x^2)] \qquad (3-3)$$

If the expected value of the gamble is denoted by

$$\mu \equiv E(x)$$

and the variance of the gamble defined by

$$\sigma^2 \equiv E(x^2) - [E(x)]^2 = E(x^2) - \mu^2$$

Then $(3 - 3)$ becomes

$$[\alpha - 2\beta W]\mu - \beta\mu^2 - \beta\sigma^2 = 0 \qquad (3-4)$$

This is the nice feature of quadratic preferences. Equation $(3-4)$ is a relation between the mean and the variance of the distribution. Everything this person needs to know about any risky proposition can be summarized by the mean and the variance.

Equation $(3-4)$ says that the higher the variance of the distribution of the returns, the higher the expected value μ of the return has to be, in order to get this risk–averse investor to invest. Conversely, it expresses the maximum variance she is willing to tolerate, as an increasing function of the expected return μ.

The equation also shows how her willingness to undertake the risky gamble varies with the person's wealth. Holding μ constant, equation $(3-4)$ shows that an increase in W must **decrease** σ^2 : for a given expected return, the higher a person's wealth is, the **lower** the amount of risk (measured by the variance) that she is willing to accept. That is, increasing W shrinks the set of gambles which the person is willing to take.

This seems somewhat unrealistic, and is an unfortunate property of quadratic utility functions. When $U(W) = \alpha W - \beta W^2$, the coefficient of absolute risk aversion increases with the person's wealth, which explains why she then becomes less willing to accept some gambles.

4. Let I denote the quantity of insurance coverage she purchases, at a total cost of pI. Her expected utility from a purchase of I dollars worth of insurance is

$$(1-\pi)\ln(W-pI) + \pi\ln(E-L+I-pI) \qquad (4-1)$$

She wants I to maximize her expected utility. So, differentiating expression $(4-1)$ with respect to I implies that

$$-\frac{p(1-\pi)}{W-pI} + \frac{(1-p)\pi}{W-L+(1-p)I} = 0 \qquad (4-2)$$

or

$$p(1-\pi)(W-L+(1-p)I) = \pi(1-p)(W-pI) \qquad (4-3)$$

which simplifies to

$$(p-\pi)W - p(1-\pi)L + p(1-p)I = 0 \qquad (4-4)$$

meaning that she will choose to purchase a quantity

$$I = (\frac{1-\pi}{1-p})L - (\frac{p-\pi}{p(1-p)})W \qquad (4-5)$$

of insurance.

Equation $(4-5)$ indicates that i she will purchase full insurance if the odds are actuarially fair (that is, if $p = \pi$) ii she will purchase less than full insurance if $p > \pi$ iii the amount of insurance she chooses to purchase is a **decreasing** function of her wealth when $p > \pi$. This last observation is consistent with the fact that her coefficient of absolute risk aversion decreases with her wealth when her utility–of–wealth function is $\ln W$.

5. Straightforward differentiation yields

$$f_1(x_1,x_2) = A - (1+\gamma)b[\frac{x_1}{x_2}]^{\gamma}$$

$$f_2(x_1,x_2) = \gamma b[\frac{x_1}{x_2}]^{1+\gamma}$$

where f_i denotes the marginal product of input i.

The $MRTS$ is just the ratio of the marginal products, so that

$$MRTS = \frac{f_1(x_1,x_2)}{f_2(x_1,x_2)} = (\frac{x_2}{x_1})\big[\frac{A}{\gamma b}(\frac{x_2}{x_1})^{\gamma} - \frac{1+\gamma}{\gamma}\big]$$

If the marginal product of both inputs is positive (which must be the case given the parameter restrictions mentioned in the question), then the $MRTS$ falls as we move down an isoquant. (That is, f_1/f_2 falls as x_1 increases and x_2 decreases.)

The production function can also be written

$$f(x_1,x_2) = x_1\big[A - b(\frac{x_1}{x_2})^{\gamma}\big]$$

which shows that it must exhibit constant returns to scale. Increasing both x_1 and x_2 by a factor t leaves (x_1/x_2) unchanged, and so must increase the value of f by a factor of t.

Note that both of the marginal products, f_1 and f_2, are homogeneous of degree 0 in x_1 and x_2. Constant returns to scale means homogeneity of degree 1 of the production function, and derivatives of a function which is homogeneous of degree k are themselves homogeneous of degree $k-1$.

The matrix H of second derivatives of this production function is

$$H = \begin{pmatrix} -\gamma(1+\gamma)b(x_1)^{\gamma-1}(x_2)^{-\gamma} & \gamma(\gamma+1)b(x_1)^{\gamma}(x_2)^{-\gamma-1} \\ \gamma(\gamma+1)b(x_1)^{\gamma}(x_2)^{-\gamma-1} & -\gamma(\gamma+1)b(x_1)^{\gamma}(x_2)^{-\gamma-2} \end{pmatrix}$$

H has negative elements on the diagonal, and a determinant of zero, showing that the production function is concave, and homogeneous of degree 1 in the inputs.