GS/ECON 5010 Answers to Assignment 3 November 2004

Q1. Suppose that firms in a competitive industry were not identical. Instead, there is one firm of each type. The cost function for a firm of type t is

$$TC(q; t) = tq + q^2$$

where q is the firm's total production, and t is its type. There is one firm of type 1, 1 of type 2, and so on. Firms are free to enter and exit the industry. What is the equation of the long–run supply curve for the industry?

A1. Given the total cost function, the marginal and average cost functions of a firm of type t are

$$MC(q; t) = t + 2q \tag{1-1}$$

$$AC(t; q) = t + q \tag{1-2}$$

Suppose that the price is p. Any firm of type $t < p$ can make a positive profit in this industry, by choosing an output level of $p - t$ or less, since it will make a profit if and only if $p > AC$. On the other hand, a firm of type $t \geq p$ cannot make a positive profit, since $AC \geq p$ for any positive output level q.

Therefore, if the price is p, then the firms which will produce positive levels of output are all those firms of type $p < n_p$, where n_p is the largest integer less than or equal to p. (So if $p = 6.3$, firms of type $1, 2, \ldots, 6$ will all produce poitive levels of output, and firms $7, 8, \ldots$ will not enter.)

Given that a firm is of type $t < n_p$, the output level it chooses is the output q for which $p = MC$. From equation $(1-1)$, $p = MC$ when

$$q = \frac{p - t}{2}$$

Here each firm's marginal and average cost curves slope up, so that $p = MC$ completely determines the firm's supply function, when $t < n_p$.

That says that the industry supply function is the sum of the supply functions of the n_p firms who choose positive output levels.

$$S(p) = \sum_{t=1}^{n_p} \frac{p - t}{2} \tag{1-3}$$

Now, for any integer n, the sum of all numbers less than or equal to n is $n(n + 1)/2$.

That means that

$$\sum_{t=1}^{n_p} t = \frac{n_p(n_p + 1)}{2}$$

meaning that

$$S(p) = \frac{pn_p}{2} - \frac{n_p(n_p + 1)}{4} \tag{1-4}$$

is the equation of the industry supply curve.

This curve slopes up, with a slope of $n_p/2$. It is also continuous. At integer prices, where sudenly a new marginal firm enters the industry, n_p jumps discontinuously. (For example, n_p jumps from 5 to 6 as p increases from 5.99999 to 6.00001.) At $p = n_p$, the value of $S(p)$ goes from

$$\frac{p(p-1)}{2} - \frac{p(p-1)}{4} \qquad (1-5)$$

to

$$\frac{p^2}{2} - \frac{p(p+1)}{4} \qquad (1-6)$$

as n_p jumps from $p-1$ to p. But expressions $(1-5)$ and $(1-6)$ both equal $p(p-1)/4$, so that the supply function is continuous. (The economics? The marginal firm, of type n_p, produces an output which goes to 0 as $p \to n_p$.)

Q2. What output would a single–price monopoly choose to produce, if it had a cost function $C(q) = cq$, faced an inverse demand function $p = a - bq$ (where a, b and c are positive constants, with $a > c$), but were also subject to a government–imposed price ceiling, that the price it charge not exceed $\bar{p}$, where $c < \bar{p} < a$?

A2. Consider the monopoly's revenue, as a function of its output q. If $p(q) > \bar{p}$, then the price ceiling is binding, and the firm's revenue is

$$R(q) = \bar{p}q \qquad (2-1)$$

On the other hand, if $p(q) < \bar{p}$, then the price ceiling is not binding, and revenue is

$$R(q) = p(q)q = aq - bq^2 \qquad (2-2)$$

The quantity level at which $p(q) = \bar{p}$ is the level for which $a - bq = \bar{p}$, or

$$\tilde{q} = \frac{a - \bar{p}}{b}$$

So equation $(2-1)$ applies for quantity levels less than $\tilde{q}$, and equation $(2-1)$ applies for quantities greater than $\tilde{q}$. Taking derivatives

$$MR(q) = \bar{p} \quad \text{if} \quad q < \tilde{q} \qquad (2-3)$$

$$MR(q) = a - 2bq \quad \text{if} \quad q > \tilde{q} \qquad (2-4)$$

Note that marginal revenue drops discontinuously at $q = \tilde{q}$, from $\bar{p} = a - b\tilde{q}$ to $a - 2b\tilde{q}$.

Now let q^M denote the "ordinary" output of a single–price monopoly, that is the output for which $MR = MC$. Here

$$q^M = \frac{a - c}{2b} \qquad (2-5)$$

and p^M the corresponding price

$$p^M = a - bq^M = \frac{a+c}{2} \tag{2 – 6}$$

If $\bar{p} > p^M$, then the price ceiling does not bind ; the monopoly will simply choose its unconstrained optimal choice of an output of q^M and a price of p^M.

What about the more interesting case, in which $\bar{p} < p^M$? If $\tilde{q} > q^M$, then the $MR = \bar{p} > c$ for $q < \tilde{q}$, and $MR = a - 2bq < a - 2bq^M = c$ for $q > \tilde{q}$. So the monopoly's marginal cost curve passes through the gap in the marginal revenue curve at $q = \tilde{q}$, and the monopoly will want to choose an output level of $\tilde{q} > q^M$, leading to a price of $\bar{p} < p^M$.

But $\bar{p} = p^M$ if and only if $\bar{p} = (a+c)/2b$, which happens if and only if

$$\tilde{q} = \frac{a - \bar{p}}{b} = \frac{a+c}{2b} = p^M$$

Thus whenever $\bar{p} < p^M$, the price ceiling will ower the monopoly's price and raise its output. (And whenever $\bar{p} > p^M$ the price ceiling has no effect.)

Q3. A market contains 1 million identical consumers, each of whom has preferences which can be represented by the utility function

$$U(X, q_1, q_2) = X + 24(q_1 + q_2) - 2[(q_1)^2 + q_1 q_2 + (q_2)^2]$$

where X is consumption of a numéraire good, and q_1 and q_2 are consumption of goods produced by firms #1 and #2 respectively.

If each firm has a constant marginal cost c of production, find the Nash equilibria if the firms choose quantities non–cooperatively (à la Cournot), **and** if they choose prices non–cooperatively (à la Bertrand).

A3. From the first–order conditions for utility maximization by each consumer,

$$U_X = 1 = \lambda$$

$$U_1 = 24 - 4q_1 - 2q_2) = \lambda p_1$$

$$U_2 = 24 - 2q_1 - 4q_2) = \lambda p_2$$

where I have used the fact that $p_X = 1$, and where U_i denotes the marginal utility of consumption of the output of firm i.

That means the inverse demand functions of the consumers for the products of firms 1 and 2 are

$$p_1 = 24 - 4q_1 - 2q_2 \tag{3 – 1}$$

$$p_2 = 24 - 2q_1 - 4q_2 \tag{3 – 2}$$

If they compete in quantities, then firm i chooses q_i so as to maximize

$$(p_i - c)q_i$$

where p_i is determined by equation $(3-1)$ or $(3-2)$, and where q_i is the firm's output, in millions of units. This maximization of

$$(24 - 4q_i - 2q_j - c)q_i$$

with respect to q_i implies reaction functions of

$$q_1 = 3 - \frac{c}{8} - \frac{q_2}{4} \tag{3-3}$$

$$q_2 = 3 - \frac{c}{8} - \frac{q_1}{4} \tag{3-4}$$

for the two firms. Solving for a symmetric equilibrium, in which $q_1 = q_2 = q$, these equations imply that

$$q^C = \frac{12}{5} - \frac{c}{10} \tag{3-5}$$

and an equilibrium price of

$$p^C = 24 - 4q^C - 2q^C = \frac{48 + 3c}{5} \tag{3-6}$$

If firms use prices as their strategic variables, we must find the consumers' Marshallian demands. Equations $(3-1)$ and $(3-2)$ can be written

$$\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 24 - p_1 \\ 24 - p_2 \end{pmatrix} \tag{3-7}$$

The matrix on the left side of equation $(3-7)$ is non–singular, and has a determinant of 12. Using Cramer's rule, equation $(3-7)$ can be solved for q_1 and q_2 as functions of p_1 and p_2 :

$$q_1 = \frac{24 - 8p_1 + 4p_2}{6} \tag{3-8}$$

$$q_2 = \frac{24 + 4p_1 - 8p_2}{6} \tag{3-9}$$

Equations $(3-8)$ and $(3-9)$ are the consumers' Marshallian demand functions for the two firms' goods.

If they compete in prices, each firm seeks to maximize $(p_i - c)q_i(p_1, p_2)$. Firm 1's probklem is to maximize

$$(p_1 - c)\frac{24 - 8p_1 + 4p_2}{6} \tag{3-10}$$

with respect to p_1. Setting the derivative of expression $(3-10)$ with respect to p_1 equal to zero,

$$p_1 = \frac{6 + p_2 + 2c}{4} \tag{3-11}$$

which is firm 1's reaction function when prices are the strategic variable. Similarly, firm 2's reaction is

$$p_2 = \frac{6 + p_1 + 2c}{4} \tag{3 – 12}$$

Solving for the symmetric Nash equilibrium in which $p_1 = p_2 = p^B$,

$$p^B = 2 + \frac{2c}{3} \tag{3 – 13}$$

which implies that the equilibrium quantities sold (per person) are (from equations $(3-8)$ and $(3-9)$)

$$q^B = \frac{33 - c}{9} \tag{3 – 14}$$

[Equation $(3-13)$ only makes sense when $c \leq 6$. If $c > 6$, equation $(3-13)$ would imply that $p < c$, which cannot be true ; firms would never want to charge a price below marginal cost. Whe $c > 6$, equations $(3-8)$ and $(3-9)$ imply each firm would sell a **negative** quantity if they charged a price equal to marginal cost.]

Equations $(3-6)$ and $(3-13)$ show that in this case, the equilibrium price is always higher under quantity competition, since $(48+3c)/5 > (6+2c)/3$, as long as $c < 114$. [And c couldn't be that high ; if $c > 24$, we would see nothing sold in this market, whatever the form of competition.]

Q4. What would be the equilibrium price, and aggregate quantity produced, in a market with 100 identical firms, each producing a homogeneous output, if the aggregate inverse demand function were

$$p = 13 - (\sum_{i=1}^{100} q_i)$$

and each firm had total costs of

$$TC = \begin{matrix} q_i + 16 & \text{if} & q_i > 0 \\ 0 & \text{if} & q_i = 0 \end{matrix}$$

where q_i is the quantity produced of good i, if firms all chose their output levels simultaneously?

A4. In this question, a firm will want to choose an output level of 0, unless its revenue can cover its fixed costs of 16.

So suppose that $J < 100$ firms choose to produce postitive quantities of output. If firm j is one of the firms producing positive quantities of output, then its profits are

$$(13 - Q)q_j - q_j - 16$$

Maximizing those profits with respect to q_j implies a first–order condition of

$$p'(Q)q_j + p(Q) - MC = 0 \tag{4 – 1}$$

In other words, the fixed costs of 16 do not matter for the firm's choice of output, given that it has chosen to produce a positive level of output. Since here $p'(Q) = -1$ and $MC = 1$, equation $(4-1)$ becomes

$$-q_j + (13 - Q) - 1 = 0 \qquad (4-2)$$

implying a reaction function of

$$q_j = 6 - \frac{Q_{\ j}}{2} \qquad (4-3)$$

where $Q_{\ j}$ is the output of all the other firms. In a symmetric equilibrium, in which the $J < 100$ firms all choose to produce the same output level q, then equation $(4-2)$ becomes

$$13 - Jq = q + 1$$

or

$$q = \frac{12}{J+1} \qquad (4-4)$$

If J firms ech produce an output of q, then the price is

$$p = 13 - J\frac{12}{J+1} = \frac{13+J}{J+1} \qquad (4-5)$$

So that fprofits of each of the J firms are

$$(p-1)q - 16 = [\frac{12}{J+1}]^2 - 16 \qquad (4-6)$$

[Or you could just plug in the expression for per–firm profits π^j in Cournot oligopoly from the middle of page 162 in the text, with $a = 13$, $b = 1$, $c = 1$, and subtract off the fixed costs of 16.]

Now firms will be willing to produce positive quantities of output only if they cover their fixed costs, that is, only if

$$[\frac{12}{J+1}]^2 \geq 16$$

which is equivalent to

$$J \leq 2$$

So we cannot have a Nash equilibrium in which more than 2 firms produce positive quantities of output.

One possible Nash equilibrium is for exactly 2 firms to produce positive quantities. From equation $(4-4)$, then we would have $q_1 = q_2 = 4$,and a price of 5, so that each firm would make a profit of $(5-1)(4) - 16 = 0$. Firms $3, 4, \cdots, 100$ would have no incentive to produce at positive levels,since they could not make a positive profit, so that one possible nash equilibrium is for any 2 firms to produce 4 units each, and all the rest to produce nothing.

There actually is another nash equilibrium. Suppose firm 1 acted as a monopoly. In this case, the single-price monopoly output is $(a-c)/2b = 6$. if firm 1 chose an output of 6, would firm 2 have any incentive to produce at a positive level? From equation $(4-3)$, firm 2's best reaction, should it decide to produce at a positive level, would be to produce $6 - (6/2) = 3$ units of output. That would lead to a market price of $(13 - 6 - 3 = 4$, which means that firm 2's profits would be $(4 - 1)3 - 16 = -7 < 0$. So here another Nash equilibrium would be for one firm to produce the single–price monopoly output 6, and for no other firm to produce anything. [In industrial organization, this second type of equilibrium is described sometimes as "blockaded entry", or "natural monopoly", since the monopoly can ignore its potential rivals, and still not have to worry about entry by another firm.]

Q5. Another model of duopoly is that of **von Stackelberg**, in which firms choose output levels **sequentially**. That is, firm 1 chooses its output. Firm 2 observes what output level firm 1 has chosen, and then chooses its own output level. What output levels would the 2 firms choose, if they behaved in this manner, if they both produced an identical product for which the market inverse demand function had the equation

$$p = A - B(q_1 + q_2)$$

if each firm had a total cost function

$$TC = cq_i$$

where q_i is the output level of firm i?

A5. When firms make their decisions sequentially, then the way to solve for the equilibrium is to work **backwards**, solving first for the reaction function of the firm which makes its decision last, in this case firm #2.

When firm #2 chooses its output level q_2, it has already observed firm #1's choice q_1. So its output will be its optimal reaction to its rival's choice q_1. In this case the reaction function of firm #2 is

$$q_2 = \frac{a-c}{2b} - \frac{q_1}{2} \qquad (5-1)$$

[Equation $(5-1)$ is just the general expression for a reaction function of a firm choosing quantity, when the industry inverse demand curve is $p = a - bQ$, when its constant marginal cost is c, and when its rival is producing q_1.]

Equation $(5-1)$ solves the last stage of the game, in which firm #2 chooses its output level. Now consider the first stage, in which firm #1 chooses what output q_1 it should produce, knowing that firm #2 will react to firm #1's own choice according to equation $(5-1)$. In other words, knowing equation $(5-1)$, and knowing that its own output choice will be observed by firm #2, firm #1can anticipate this reaction, and knows that if it increases its own output by 1 unit, firm #2 will react be reducing its output by 0.5 units.

Firm #1's profit is

$$(a - b(q_1 + q_2))q_1 - cq_1$$

Substituting from equation $(5-1)$ yields

$$\pi_1(q_1) = [a - bq_1 - b(\frac{a-c}{2b} - \frac{q_1}{2})]q - 1 - cq_1 \qquad (5-2)$$

or

$$\pi_1(q_1) = \frac{1}{2}[a - c - bq_1]q_1 \qquad (5-3)$$

Maximizing $(5-3)$ with respect to $q_1$yields

$$q_1^* = \frac{a-c}{2b} \qquad (5-4)$$

as firm #1's optimal choice, when it anticipates the reaction of firm #2. Plugging $(5-4)$ into $(5-1)$, firm #2 will choose an output of

$$q_2 = \frac{a-c}{4b} \qquad (5-5)$$

when firm #1 chooses its profit–maximizing output.

In the equilibrium, the price is

$$p = a - b[q_1^* + q_2(q_1^*)] = \frac{a+3c}{4b} \qquad (5-6)$$

and the profits of the two firms are

$$\pi_1 = (p-c)q_1^* = \frac{(a-c)^2}{8b} \qquad (5-7)$$

$$\pi_2 = (p-c)q_2(q_1^*) = \frac{(a-c)^2}{16b} \qquad (5-8)$$

So moving first, and committing to a high level of output, is a strategic advantage here. The firm which chooses first makes higher profits. (And the firm moving last actually makes lower profits than if both firms chose outputs simultaneously, as in the Cournot model.)