GS/ECON 5010 Answers to Assignment 1

September 2005

*Q*1. Are the preferences represented by the following utility function strictly monotonic? Convex?

$$u(x_1, x_2, x_3) = \ln(x_1^2 + x_2^2 + x_3^2)$$

In each case, explain briefly.

*A*1. In this case, it is most convenient to take a monotonic increasing transformation of the utility function. If we take the "antilog" of $u(x_1, x_2, x_3)$, that is take the function $U(\mathbf{x}) = e^{u(\mathbf{x})}$, then the preferences can also be represented by the utility function

$$U(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

These preferences are strictly monotonic, since increasing **any** component of (x_1, x_2, x_3) must increase — strictly — the value of $x_1^2 + x_2^2 + x_3^2$.

But the preferences are **not** convex. One way of showing this is by constructing a counter–example. Let S be the set of all consumption bundles which are at least as good as the bundle $(1, 0, 0)$. Since $U(1, 0, 0) = 1$, S consists of all consumption bundles $\mathbf{x}$ for which $x_1^2 + x_2^2 + x_3^2 \geq 1$. The bundles $(0, 1, 0)$ and $(0, 0, 1)$ are both in S, since $U(0, 1, 0) = U(0, 0, 1) = 1$. But now take the bundle $\mathbf{x}' \equiv (0.0.5, 0.5)$. The bundle $\mathbf{x}'$ is halfway along the line connecting the bundles $(0, 1, 0)$ and $(0, 0, 1)$. But since $U(0, 0.5, 0.5) = 0.5 < 1$, $\mathbf{x}'$ is not in S. So the set S, defined as the set of consumption bundles which were at least as good as the bundle $(1, 0, 0)$, is not a convex set. Therefore, the preferences represented by this utility function are not convex.

Another way of showing preferences are not convex is to look at the indifference curves in any two dimensions. If preferences are convex, and if they are strictly monotonic, then indifferences curves in x_1–x_2 space (holding x_3 constant) must have the usual "bowed in to the origin" shape. In this case, however, the indifference curves get steeper as we move down and to the right, so that preferences are not convex. [1]

The fact that preferences are not convex can be shown also be showing that the principal minors of the bordered Hessian matrix do not alternate in sign.

[1] Unfortunately, the above necessary condition is not sufficient. That is, indifference curves representing some preferences in any two dimensions could have the right, "bowed–in" shape, and the preferences still might not be convex.

*Q*2. Are the preferences represented by the following utility function strictly monotonic? Convex?

$$u(x_1, x_2, x_3) = -(\frac{1}{x_1^1} + \frac{1}{x_2^2} + \frac{1}{x_3^3})$$

In each case, explain briefly.

*A*2. Since the partial derivatives,

$$\frac{\partial u}{\partial x_1} = \frac{1}{x_1^2} \quad \frac{\partial u}{\partial x_2} = \frac{2}{x_2^3} \quad \frac{\partial u}{x_3} = \frac{3}{x_3^4}$$

are all stictly positive (when $\mathbf{x} >> 0$), the preferences are strictly monotonic.

The matrix of second derivatives of the utility function $u(\mathbf{x})$ is

$$H \equiv \begin{pmatrix} -2x_1^{-3} & 0 & 0 \\ 0 & -6x_2^{-4} & 0 \\ 0 & 0 & -12x_3^{-5} \end{pmatrix}$$

H is a negative definite matrix : negative numbers on the diagonal, and zeroes off the diagonal. Therefore the function $u(\mathbf{x})$ is a concave function, which implies that the preferences represented by the utility function $u(\mathbf{x})$ must be convex.

The convexity of these preferences can also be seen by showing that $\mathbf{v}^T H\mathbf{v} \leq 0$ for any direction $\mathbf{v}$. That's true because here

$$\mathbf{v}^T H\mathbf{v} \leq 0 = -2(x_1)^{-3}(v_1)^2 - 6(x_2)^{-4}(v_2)^2 - 12(x_3)^{-5}(v_3)^2$$

which must be non–positive whenever $\mathbf{x} \geq \mathbf{0}$.

*Q*3. What would a person's Marshallian demand functions be if her preferences could be represented by the following utility function?

$$u(x_1, x_2, x_3) = x_1 + \min(x_2, x_3)$$

*A*3. Because of the $\min(x_2, x_3)$ term, the person will always choose a consumption bundle for which $x_2 = x_3$: if $x_2 < x_3$ she could spend a little more money on good 2, a little less on good 3, and increase the value of her utility without spending any more money. (If $x_3 < x_2$, she would want to do the opposite.)

That means that her utility maximization can be reduced to a two–dimensional problem. She should pick a level x_1 of consumption of good 1, and then pick some $x_2 = x_3 = z$, with $p_1x_1 + (p_2 + p_3)z$.

But that is exactly the problem of choosing between two goods which are perfect substitutes, one of which has the price p_1, and the other of which has the price $p_2 + p_3$.

So whenever $p_1 > p_2 + p_3$ she should spend nothing on good 1, and whenever $p_1 < p_2 + p_3$ she should spend all her money on good 1. Her Marshallian demand functions are :

$$\mathbf{x}^M(\mathbf{p}, y) = (0, \frac{y}{p_2 + p_3}, \frac{y}{p_2 + p_3}) \qquad \text{if} \quad p_1 > p_2 + p_3$$

$$\mathbf{x}^M(\mathbf{p}, y) = (\frac{y}{p_1}, 0, 0)) \qquad \text{if} \quad p_1 < p_2 + p_3$$

If $p_1 = p_2 + p_3$ her Marshallian demands are not unique (preferences here are not **strictly** convex) ; any consumption bundle (x_1, z, z) with $x_1 \geq 0$, $z \geq 0$ and $p_1 x_1 + (p_2 + p_3)z = y$ would be a Marshallian demand vector if $p_1 = p_2 + p_3$.

*Q*4. If a person's preferences can be represented by the direct utility function

$$U(x_1, x_2, x_3) = 2\sqrt{x_1} + 2\sqrt{(x_2 x_3)}$$

find her Marshallian demand functions, and her indirect utility function.

*A*4. First–order conditions here for utility maximization are

$$\frac{\partial U}{\partial x_1} - \lambda p_1 = \frac{1}{\sqrt{x_1}} - \lambda p_1 = 0 \tag{1}$$

$$\frac{\partial U}{\partial x_2} - \lambda p_2 = \sqrt{\frac{x_3}{x_2}} - \lambda p_2 = 0 \tag{2}$$

$$\frac{\partial U}{\partial x_3} - \lambda p_3 = \sqrt{\frac{x_2}{x_3}} - \lambda p_3 = 0 \tag{3}$$

But equations (2) and (3) together imply that

$$\frac{x_2}{x_3} = \frac{p_3}{p_2} \tag{4}$$

So that

$$x_2 = \frac{p_2}{p_3} x_3 \tag{5}$$

Now use equation (5) to substitute for x_3 in the original utility function, so that utility can now be written

$$2\sqrt{x_1} + 2\sqrt{\frac{p_2}{p_3}} x_2 \tag{6}$$

If we maximize expression (6) with respect to x_1 and x_2, subject to the budget constraint

$$y = p_1 x_1 + p_2 x_2 + p_3 \frac{p_2}{p_3} x_2 = p_1 x_1 + 2 p_2 x_2 \tag{7}$$

we get new first–order conditions

$$\frac{1}{\sqrt{x_1}} = \mu p_1 \tag{8}$$

$$2\sqrt{\frac{p_2}{p_3}} = 2\mu p_2 \tag{9}$$

where μ is the Lagrange multiplier associated with the "new" constraint (7)). Equation (9) implies that $\mu = \frac{1}{\sqrt{p_2 p_3}}$. Substituting for μ in equation (8) yields the Marshallian demand function for good 1,

$$x_1^M(\mathbf{p}, y) = \frac{p_2 p_3}{(p_1)^2} \tag{10}$$

Substituting for x_1 in equation (7) yields the Marshallian demand function for good 2,

$$x_2^M(\mathbf{p}, y) = \frac{y}{2p_2} - \frac{p_3}{p_1} \tag{11}$$

and substituting from (11) into (4) yields the Marshallian demand function for good 3,

$$x_3^M(\mathbf{p}, y) = \frac{y}{2p_3} - \frac{p_2}{p_1} \tag{12}$$

(These demands hold only when $yp_1 > (p_2 p_3)$; if $yp_1 < p_2 p_3$, then the consumer spends all her money on good 1 : $x_1^M(\mathbf{p}, y) = y/p_1$ and $x_2^M(\mathbf{p}, y) = x_3^M(\mathbf{p}, y) = 0$.)

Substituting from (10) and (11) back into (6), her indirect utility function is

$$v(\mathbf{p}, y) = 2\frac{\sqrt{p_2 p_3}}{p_1} + 2\sqrt{\frac{p_2}{p_3}}[\frac{y}{p_2} - \frac{p_3}{2p_1}] = \frac{\sqrt{p_2 p_3}}{p_1} + \frac{y}{\sqrt{p_2 p_3}} \tag{13}$$

*Q*5. Find a person's Hicksian (compensated) and Marshallian (uncompensated) demand functions if her expenditure function can be written

$$e(p_1, p_2, u) = p_1 u - \frac{(p_1)^2}{p_2}$$

(if $u > 2p_1/p_2$).

*A*5. The Hicksian demand functions are just the partial derivatives (with respect to the prices) of the expenditure, so that

$$x_1^H(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_1} = u - 2\frac{p_1}{p_2}$$

$$x_2^H(p_1, p_2, u) = \frac{\partial e(p_1, p_2, u)}{\partial p_2} = \frac{(p_1)^2}{(p_2)^2}$$

To find the Marshallian demands, first find the indirect utility function. The definition of the expenditure function implies that

$$p_1 v(p_1, p_2, m) - \frac{(p_1)^2}{p_2} = y$$

or

$$v(p_1, p_2, m) = \frac{y}{p_1} + \frac{p_1}{p_2}$$

Roy's identity says that the Marshallian demand function for a good equals the negative of the partial derivative of the indirect utility function with respect to the price of the good divided by the derivative of the indirect utility function with respect to income. Here

$$\frac{\partial v(p_1, p_2, y)}{\partial y} = \frac{1}{p_1}$$

$$\frac{\partial v(p_1, p_2, y)}{\partial p_1} = -\frac{y}{(p_1)^2} + \frac{1}{p_2}$$

$$\frac{\partial v(p_1, p_2, y)}{\partial p_2} = -\frac{p_1}{(p_2)^2}$$

so that

$$x_1^M(p_1, p_2, y) = \frac{y}{p_1} - \frac{p_1}{p_2}$$

$$x_2^M(p_1, p_2, y) = \frac{(p_1)^2}{(p_2)^2}$$

Notice that here the Marshallian and Hicksian demand functions for good 2 are identical, since the Marshallian demand for good 2 is independent of income.