$Q1$. The following table lists the prices of 2 goods, and the quantities a consumer chose of the goods, in 5 different situations. (For example, the second row indicates that the consumer chose the bundle $\mathbf{x} = (10, 32)$ when the price vector was $\mathbf{p} = (4, 2)$.)

From these data, what can be concluded about the consumer's preferences? Explain briefly.

t	p_1^t	p_2^t	x_1^t	x_2^t
1	5	1	5	40
2	4	2	10	32
3	3	3	11	5
4	2	4	20	11
5	1	5	25	3

$A1$. Given the data in the table above, the cost of each bundle in each situation can be computed. Rows here represent the different periods, and columns the different bundles.

t	bundle1	bundle2	bundle3	bundle4	bundle5
1	65	82	60	111	128
2	100	104	54	102	106
3	135	126	48	93	84
4	170	148	42	84	62
5	205	170	36	75	40

(So, for example, bundle 2 would cost 148 using period–4 prices.)

If an element on the diagonal of the above matrix is greater than or equal to an element in the same row, then the bundle chosen in that year has been revealed preferred to the bundle chosen in the other year.

So, for example, the first row of the matrix shows that $\mathbf{x}^1$ was chosen when $\mathbf{x}^3$ was in the budget set, so that $\mathbf{x}^1$ is revealed preferred to $\mathbf{x}^3$. Using the shorthand "rpt" for "revealed preferred to" (or "chosen over"), the rows show that $\mathbf{x}^1$ rpt $\mathbf{x}^{\mathbf{3}}$; $\mathbf{x}^2$ rpt all of $\mathbf{x}^1$, $\mathbf{x}^3$ and $\mathbf{x}^4$; $\mathbf{x}^4$ rpt $\mathbf{x}^3$ and $\mathbf{x}^5$; $\mathbf{x}^5$ rpt $\mathbf{x}^3$.

This consumer's preferences do not violate the weak axiom of revealed preference. The data also show that bundle 2 is her most preferred bundle, and bundle 3 her least preferred. Bundles 1, 4 and 5 are ranked in the middle : the only information given by the table concerning the relative ranking of these three bundles is that 4 is ranked higher than 5. But it would be consistent with these data for her to rank them $1 \succeq 4 \succeq 5$, or $4 \succeq 1 \succeq 5$, or $4 \succeq 5 \succeq 1$.

[If you did the original "mistaken" version of the question, in which the prices in situation #5 were $(5, 1)$, then there would be 2 violations of $WARP$: bundles 2 and 5, and bundles 4 and 5.]

$Q2$. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at four different times, and the quantities x^t of the 3 goods chosen at the four different times.

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	5	2	7	6	10	12
2	8	4	2	5	12	15
3	10	2	4	10	10	10
4	2	10	2	8	12	10

$A2$.. Again a matrix can be constructed showing the cost of each bundle in each period.

t	bundle1	bundle2	bundle3	bundle4
1	134	154	140	134
2	112	118	140	132
3	128	134	160	144
4	136	160	140	156

Letting "irpt" stand for "'indirectly revealed preferred to", then the rows show : bundle 1 rpt bundle 4 ; bundle 2 rpt bundle 1 ; bundle 3 rpt all other bundles ; bundle 4 rpt bundles 1 and 3.

So : bundle 1 rpt bundle 4 but bundle 4 rpt bundle 1 ; bundle 3 rpt bundle 4 but bundle 4 rpt bundle 3 ; bundle 1 irpt bundle 2 (since bundle 1 rpt bundle 4 rpt bundle 3 rpt bundle 2) but bundle 2 rpt bundle 1 ; bundle 1 irpt bundle 3 (since bundle 1 rpt bundle 4 rpt bundle 3) but bundle 3 rpt bundle 1 ; bundle 2 irpt bundle 3 (since bundle 2 rpt bundle 1 rpt bundle 4 rpt bundle 3) but bundle 3 rpt bundle 2 ; bundle 2 irpt bundle 4 (since bundle 2 rpt bundle 1 rpt bundle 4) but bundle 4 irpt bundle 2 (since bundle 4 rpt bundle 3 rpt bundle 2).

There are 2 pairs of consumption bundles which violate $WARP$, and another 4 which violate $SARP$ but not $WARP$. In fact here every pair of consumption bundles violates $SARP$.

$Q3$. Suppose that a person's utility–of–wealth function could be written

$$u(W) = A - e^{-\beta W}$$

where $\beta > 0$.

What would be the risk premium associated with a project which yielded the person a return of $X > 0$ with probability π, and a payoff of zero with probability $1 - \pi$? How does the premium vary with the "good state" return X?

$A3$. The certainty equivalent C to this project is the solution to the equation

$$e^{-\beta(W+C)} = \pi e^{-\beta(W+X)} + (1-\pi)e^{-\beta W} \qquad (3-1)$$

where the left side is the utitlity from geting C for sure, and the right side is the expected utility from the project.

Dividing both sides of equation $(3-1)$ by $e^{-\beta W}$,

$$e^{-\beta C} = \pi e^{-\beta X} + 1 - \pi \qquad (3-2)$$

or

$$C = -\frac{1}{\beta} \ln\left(\pi e^{-\beta X} + 1 - \pi\right) \qquad (3-3)$$

The risk premium P equals the expected return to the project, minus C. The expected return here is πX, so that

$$P = \pi X + \frac{1}{\beta} \ln\left(\pi e^{-\beta X} + 1 - \pi\right) \qquad (3-4)$$

Differentiating $(3-4)$,

$$\frac{\partial P}{\partial X} = \pi - \frac{\pi e^{-\beta X}}{\pi e^{-\beta X} + 1 - \pi} \qquad (3-5)$$

Now since $X > 0$, then

$$e^{-\beta X} < 1$$

so that

$$\pi e^{-\beta X} + 1 - \pi > \pi e^{-\beta X} + (1-\pi) e^{-\beta X} = e^{-\beta X} \qquad (3-6)$$

implying that

$$\frac{\pi e^{-\beta X}}{\pi e^{-\beta X} + 1 - \pi} < \pi \qquad (3-7)$$

Equation $(3-7)$ (substituted into equation $(3-5)$) indicates that an increase in the return X when the investment pays off will increase the risk premium P associated with the investment.

*Q*4. Suppose a person's utility–of–wealth function could be written

$$u(W) = W^a$$

where $0 < a < 1$.

Suppose as well that the person had to choose between investing all her initial wealth in a bond, which gave a certain return of r_0, and putting all her initial wealth in a risky asset, the gross return $1 + r$ for which was distributed uniformly over the interval $[0, R]$. (That is, if she put all her wealth W_0 in the risky asset, her end–of–period wealth would be distributed uniformly over $[0, RW_0]$.)

What value of R would make her indifferent between putting all her wealth in the safe asset, and all her wealth in the risky asset? How does this R vary with her initial wealth, with her risk aversion parameter a, and with the gross return $1 + r_0$ on the safe asset?

A4. The expected utility that the person gets if she puts all her money in the bond is

$$[W_0(1+r_0)]^a$$

whereas if she put all her money in the risky asset, her expected utility would be

$$\frac{1}{R}\int_0^R [W_0X]^a dX$$

(since the density function of a random variable which is uniformly distributed over $[0, R]$ is $f(x) = 1/R$).

Therefore, she will be indifferent between putting all her money in the risky asset, and putting all her money in the safe asset if and only if

$$[W_0(1+r_0)]^a = \frac{1}{R}\int_0^R [W_0X]^a dX$$

or

$$(1+r_0)^a = \frac{1}{R}\int_0^R X^a dX \tag{4-1}$$

Solving the integral in equation $(4-1)$,

$$(1+r_0)^a = \frac{1}{R}\frac{1}{1+a}R^{1+a} \tag{4-2}$$

implying that

$$R = (1+a)^{1/a}(1+r_0) \tag{4-3}$$

The maximum return R which satisfies $(4-3)$ is proportional to the gross return $1+r_0$ on the safe asset. It is independent of the person's initial wealth W_0.

This person's utility–of–wealth function implies that she has a constant coefficient of relative risk aversion,

$$R_r(W) = -\frac{u''(W)W}{u'(W)} = 1-a \tag{4-4}$$

The higher is a person's coefficient of relative risk aversion, the less likely she is to accept gambles. That means that the value of R which satisfies equation $(4-3)$ must be a **decreasing** function of $a:$ if a person with utility parameter a were indifferent between the safe asset and the risky asset, then a person with a lower value of the parameter, $\tilde{a} < a$ would require a higher expected return on the risky asset in order to put her money in the risky asset.

To prove directly how R varies with a, note (from $(4-3)$) that R increases with a if and only if $(1+a)^{1/a}$ increases with a. To see whether $(1+a)^{1/a}$ increases with a, take the logarithm of this function — $(1+a)^{1/a}$ will increase with a if and only if $\ln[(1+a)^{1/a}]$ increases with a. That logarithm is

$$f(a) \equiv \frac{1}{a}\ln(1+a)$$

Differentiating,

$$f'(a) = \frac{1}{a^2}[\frac{a}{1+a} - \ln(1+a)] \qquad (4-5)$$

So $f'(a) > 0$ if and only if

$$g(a) \equiv \frac{a}{1+a} - \ln(1+a)$$

is positive. Now $g(0) = 0$, and

$$g'(a) = -\frac{a}{(1+a)^2} < 0 \qquad (4-6)$$

So $g(a) < 0$ for all $a \geq 0$, implying that $f(a)$ must be a decreasing function of a (for all $a > 0$). Therefore the required maximum gross return R on the risky asset is a **decreasing** function of a, and an increasing function of the person's coefficient of relative risk aversion.

Note that the expected gross return on the risky asset is $R/2$. So if $a = 1$, equation $(4-3)$ implies that $1 + r_0 = R/2$, so that both assets have the same expected return : not surprising, since $a = 1$ implies that the person is risk neutral. Since R is a decreasing function of a, a positive coefficient of relative risk aversion implies that $R/2 > 1 + r_0$: the risk premium associated with the risky asset is positive, and increases with the person's coefficient of relative risk aversion.

*Q*5. If a production function $f(x_1, x_2)$ has the equation

$$f(x_1, x_2) = [x_1 \ln(\frac{x_1 + x_2}{x_1})]^a$$

where $0 < a < 1$, calculate the marginal product of each input, and the marginal rate of technical substitution. Does the production function exhibit decreasing, constant, or increasing returns to scale? Explain briefly.

*A*5. Brute force differentiation of the definition of the production function yields :

$$f_1(x_1, x_2) = a[x_1 \ln(\frac{x_1 + x_2}{x_1})]^{a-1}[\ln(\frac{x_1 + x_2}{x_1}) - \frac{x_2}{x_1 + x_2}] \qquad (5-1)$$

$$f_2(x_1, x_2) = a[x_1 \ln(\frac{x_1 + x_2}{x_1})]^{a-1}\frac{x_1}{x_1 + x_2} \qquad (5-2)$$

as the marginal products of the two inputs.

Therefore, the marginal rate of technical substitution is

$$MRTS = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{x_1 + x_2}{x_1} \ln(\frac{x_1 + x_2}{x_1}) - \frac{x_2}{x_1} \qquad (5-3)$$

To see how the marginal rate of technical substitution varies as we move along an isoquant, let

$$g(z) \equiv (1 + z)\ln(1 + z) - z$$

so that equation $(5-3)$ implies that

$$MRTS = g(\frac{x_2}{x_1}) \qquad (5-4)$$

Since

$$g'(z) = \ln(1 + z) > 0$$

therefore the marginal rate of substitution increases as we increase z_2 and lower z_1 : the isoquants are convex to the origin, and the production function $f(x_1, x_2)$ is quasi–concave.

What would happen if both x_1 and x_2 were to increase by some factor k? The defintion of the production function implies that

$$f(kx_1, kx_2) = [kx_1 \ln(\frac{kx_1 + kx_2}{kx_1})]^a = [kx_1 \ln(\frac{x_1 + x_2}{x_1})]^a = k^a[x_1 \ln(\frac{x_1 + x_2}{x_1})]^a = k^a f(x_1, x_2) \tag{5 – 5}$$

Therefore, the production function is homogeneous of degree $a < 1$: the production function exhibits decreasing returns to scale.

Since the production function is quasi–concave, and exhibits decreasing returns to scale, then it must be concave. (This is really a consequence of Theorem 3.1 in the book, that quasi–concave production functions must be concave if they exhibit constant returns to scale.) Concave functions must have non–negative elements along the diagonal of the Hessian matrix. Therefore $f_{11} \leq 0$ and $f_{22} < 0$, so that the marginal product of each input decreases with the quantity employed of that input, a property which may not be immediately apparent from equations $(5 - 1)$ and $(5 - 2)$.