GS/ECON 5010 Answers to Assignment 3 November 2005

*Q*1. What are the market price, and aggregate quantity sold, in long–run equilibrium in a perfectly competitive market for which the demand function has the equation

$$Q = \frac{6000000}{p}$$

(where Q is aggregate quantity demanded, and p the price), if there is free entry by identical firms to the industry, each of which has the long–run total cost function

$$TC = 9000q - 600q^2 + 15q^3$$

where q is the quantity produced by the firm?

*A*1. Given that there is free entry by identical firms, in long–run equilibrium each firm in the industry will choose an output level at the bottom of its average cost curve, where $MC = AC$. (Only this level of output is consistent with both profit maximization by each firm, and zero profits due to free entry by identical firms.)

Given the total cost function,

$$AC(q) = 9000 - 600q + 15q^2 \qquad (1-1)$$

and

$$MC(q) = 9000 - 1200q + 45q^2 \qquad (1-2)$$

so that $MC(q) = AC(q)$ if

$$600 = 30q$$

or

$$q = 20$$

If a firm produces an output level of 20, then its average cost (and its marginal cost) is

$$9000 - 600(20) + 15(20)^2 = 3000 \qquad (1-2)$$

Therefore, the industry long–run supply curve is horizontal, at a height of 3000.

At a price of 3000, aggregate demand is $6000000/3000 = 2000$. So in long–run equilibrium, there will be 100 firms in the industry, each producing an output of 20, at an average (and marginal) cost of 3000, which is the equilibrium price.

*Q*2. Suppose that firms in a competitive industry were not identical. Instead, there are 10 firms each with a cost function $TC(q) = q^2/2$, 10 more firms each with a total cost function $TC(q) = q + q^2/2$, 10 more firms each with a total cost function of $TC(q) = 2q + q^2/2$, another 10 firms each with cost function $TC(q) = 3q + q^2/2$, and so on. Firms are free to enter and exit the industry. What is the equation of the long–run supply curve for the industry?

*A*2. A "type i" firm will refer to a firm which has a cost function of $iq + q^2/2$. (So there are 10 firms of type 0, 10 of type 1, 10 of type 2, and so on.) A type i firm has

$$AC^i(q) = i + \frac{q}{2} \tag{2 − 1}$$

$$MC^i(q) = i + q \tag{2 − 2}$$

Equation $(2 - 1)$ says that a firm of type i has an upward–sloping average cost function, which starts with $AC = i$ at $q = 0$. So the type–i firm can make a non–negative profit in this industry if (and only if) the market price is at least as large as its minimum average cost, i.

So, for example, if $p = 3.3$, then all firms of type 0,1,2 and 3 can make positive profits. But any firms of type $4, 5, 6, \ldots$ cannot make a positive profit. They will not enter the industry.

Since there are 10 firms of each type, that means that if the market price of the good is between j and $j + 1$, then there will be $10(j + 1)$ firms in the industry. A higher price induces entry by more firms : the higher–type firms which can only make a profit if the price is high.

If a firm chooses to produce a positive level of output, how much should it produce? It wil maximize price by choosing an output level q such that $MC = p$. So if $p > i$, then a firm of type i will choose an output level such that $i + q = p$, or

$$q = p - i \tag{2 − 3}$$

So, if the price of the good were p, with $j < p < j + 1$, the total quantity supplied by the $10(j + 1)$ firms in the industry would be

$$Q(p) = 10p + 10(p - 1) + 10(p - 2) + \cdots + 10(p - j) = 10(j + 1)p - 10\frac{j(j + 1)}{2} \tag{2 − 4}$$

This industry supply curve is continuous : at p=3, for example, 10 new firms enter the industry (the firms of type 3), but they each produce an output which approaches 0 as p approaches 3 from above. The industry supply curve also gets less steep as the price rises here, since

$$\frac{\partial Q}{\partial p} = 10(j + 1) \tag{2 − 5}$$

if $j < p < j + 1$.

*Q*3. How would the output of a single–price monopoly vary with its fixed cost F, if it had a cost function $C(q) = F + cq$, and faced an inverse demand function $p = a - bq$ (where a, b and c are positive constants, with $a > c$)?

*A*3. The monopoly's profit, as a function of its output q, is

$$\pi(q) = (a - bq)q - (F + cq) \tag{3 − 1}$$

The first–order condition for profit maximization is

$$q = \frac{a-c}{2b} \tag{3 - 2}$$

so that a profit maximizing firm should pick a quantity level of $q = (a-c)/2b$. (The second–order condition, that this is a local maximum and not a minimum, is satisfied here.)

What is the firm's profit? Substituting from $(3-2)$ into $(3-1)$, its profit is

$$(a - b\frac{a-c}{2b})\frac{a-c}{2b} - F - c\frac{a-c}{2b}$$

which equals

$$\frac{(a-c)^2}{4b} - F \tag{3 - 3}$$

So the firm makes a positive profit only if $(a-c)^2/4b \geq F$. If F is large enough that expression $(3-3)$ is negative, then it cannot cover its fixed costs. It is better off producing nothing. Therefore, the firm's output does vary, in a very crude way, with its costs : if $F > (a-c)^2/4b$ its optimal output is 0, and if $F < (a-c)^2/4b$ then its optimal output is $(a-c)/2b$. (If $F = (a-c)^2 4b$ then both $q = 0$ and $q = (a-c)/2b$ would yield it profits of 0, so that it would have two optimal choices.)

*Q*4. What is the Cournot equilibrium, if there are $n > 1$ firms in the industry, each producing a homogeneous product, each with identical total cost function $TC(q) = cq$ where $c > 0$ is some constant, if the market demand function is

$$Q^D = p^{-a}$$

where p is the price of the good, Q^D the total quantity demanded, and $a > 1/n$?

*A*4. The profit of a firm producing q is

$$Q^{-1/a}q - cq \tag{4 - 1}$$

since the market price is $Q^{-1/a}$. Differentiating expression $(4-1)$ with respect to q (and recognizing that Q depends on q) yields the first–order condition

$$-\frac{1}{a}Q^{-1/a-1}q + Q^{-1/a} - c = 0 \tag{4 - 2}$$

Expression $(4-2)$ shows that all firms must have the same level of output in any Cournot–Nash equilibrium : if firm 1 chooses q_1 optimally, and if firm 2 chooses q_2 optimally, then equation $(4-2)$ says that

$$-\frac{1}{a}Q^{-1/a-1}(q_1 - q_2) = 0 \tag{4 - 3}$$

since both firms' optimal choice of output q_i must satisfy equation $(4-2)$.

So the only possible Cournot–Nash equilibrium is one in which all firms produce the same level of output, so that $Q = nq$, which means that equation $(4-2)$ implies

$$-\frac{1}{a}n^{-1/a-1}q^{-1/a} + n^{-1/a}q^{-1/a} = c \tag{4-4}$$

which means that the output of each identical firm is

$$q = c^{-a}(n - \frac{1}{a})^a n^{-(a+1)} \tag{4-5}$$

so that the industry's total output is

$$Q = c^{-a}(n - \frac{1}{a})^a n^{-a} \tag{4-6}$$

and the industry price is

$$p = \frac{an}{an-1}c \tag{4-7}$$

As in the case of linear demand, the oligopoly price is a decreasing function of the number of firms in the industry, converging to the marginal cost as the number of firms becomes large.

Now it also should be checked that the above solution actually is an equilibrium : are firms maximizing profits when they choose q satisfying equation $(4-4)$? The second–order condition for profit maximization is, from equation $(4-2)$,

$$\frac{1+a}{a}\frac{1}{a}Q^{-1/a-2}q - \frac{2}{a}Q^{-1/a-1} \tag{4-8}$$

If $Q = nq$, then expression $(4-8)$ must be proportional to

$$\frac{1}{n}\frac{1+a}{a} - 2$$

which must be negative if $a > 1/n$, so that the second–order conditions for optimality are satisfied.

*Q*5. Solve for the equilibrium of a Cournot duopoly, if firms produce a homogeneous output, the demand for which obeys the function $p = a - Q$ where p is the price of the good, $Q = q_1 + q_2$ is the total quantity sold, and $a > 0$, if firm #1 could produce the good for nothing, and if firm #2 had the total cost function

$$TC(q_2) = cq_2$$

where $a > c > 0$.

How does the profit of firm 1 vary with its rival's marginal cost c of production?

*A*5. Since the market demand is a linear function of the price, and since each firm has constant marginal costs, then the reaction curves of the 2 firms are

$$q_1 = \frac{a}{2} - \frac{q_2}{2} \tag{5-1}$$

$$q_2 = \frac{a-c}{2} - \frac{q_1}{2} \qquad (5-2)$$

Adding together equations $(5-1)$ and $(5-2)$,

$$Q = a - \frac{c}{2} - \frac{Q}{2} \qquad (5-3)$$

implying that total industry output is

$$Q = \frac{2a-c}{3} \qquad (5-4)$$

Since $q_1 = Q - q_2$, plugging equation $(5-3)$ into $(5-1)$ implies that

$$\frac{2a-c}{3} - q_2 = \frac{a}{2} - \frac{q_2}{2} \qquad (5-5)$$

or

$$q_2 = \frac{a-2c}{3} \qquad (5-6)$$

so that

$$q_1 = \frac{a+c}{3} \qquad (5-7)$$

Increases in firm 2's cost will decrease its own equilibrium output, and increase that of its rival. To calculate profits, note that the price is $a - Q$, so that in equilibrium

$$p = \frac{a+c}{3} \qquad (5-8)$$

meaning that firm 1's profit is

$$\frac{(a+c)^2}{3}$$

an increasing function of its rival's cost level c.

However, equation $(5-6)$ implies that $q_2 > 0$ if and only if $a > 2c$. If $a > c > a/2$, then firm 2 does not produce anything in equilibrium : then firm 1 produces its monopoly output of $a/2$, and firm 2's best reaction is not to produce at all. In this range, firm 1's profits do not vary with further changes in c.