*Q*1. What does the contract curve look like for a 2–person, 2–good exchange economy, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = 100 - \frac{1}{x_1^1} - \frac{1}{x_2^1}$$

$$u^2(x_1^2, x_2^2) = x_2^2 + 100 \ln x_1^2$$

where x_j^i is person i's consumption of good j?

*A*1. The two people's marginal rates of substitution are

$$MRS^1 = \frac{u_1^1}{u_2^1} = \frac{(x_2^1)^2}{(x_1^1)^2}$$

$$MRS^2 = \frac{u_1^2}{u_2^2} = \frac{100}{x_1^2}$$

So an interior allocation will be on the contract curve if $MRS^1 = MRS^2$, or

$$\frac{(x_2^1)^2}{(x_1^1)^2} = \frac{100}{x_1^2} \qquad (1-1)$$

Given that

$$x_1^2 = e_1 - x_1^1$$

where e_j is the total endowment of good j, equation $(1-1)$ can be written

$$x_2^1 = 10\frac{x_1^1}{\sqrt{e_1 - x_1^1}} \qquad (1-2)$$

which defines an upward–sloping curve in x_1^1–x_2^1 space.

The contract curve goes through the bottom left corner of the Edgeworth box : when $x_1^1 = 0$, equation $(1-2)$ indicates that $x_2^1 = 0$. However, the curve defined by equation $(1-2)$ hits the top edge of the Edgeworth box : as $x_1^1 \to e_1$, $x_2^1 \to \infty$ according to $(1-2)$.

The contract curve hits the top of the Edgeworth box when $x_2^1 = e_2$, or

$$e_2 = 10\frac{x_1^1}{\sqrt{e_1 - x_1^1}} \qquad (1-3)$$

Equation $(1-3)$ can be solved for the value of x_1^1 at which $x_2^1 = e_2$ along the contract curve. Using the quadratic formula

$$x_1^1 = \frac{e_2\sqrt{(e_2)^2 + 400e_1} - (e_2)^2}{200} \equiv \bar{x_1} \qquad (1-4)$$

So the contract curve consists of all (x_1^1, x_2^1) pairs satisfying equation $(1-2)$, with $0 \leq x_1^1 \leq \bar{x_1}$, and then the set of all (x_1^1, x_2^1) with $\bar{x_1} < x_1^1 \leq e_1$, and $x_2^1 = e_2$.

*Q*2. What are all the allocations in the core of a 3–person, 2–good economy, in which each person has the same endowment vector, $e^i = (1, 1)$, and in which the preferences of the 3 people can be represented by the utility functions listed below?

$$u^1(x_1^1, x_2^1) = x_1^1$$

$$u^2(x_1^2, x_2^2) = x_2^2$$

$$u^3(x_1^3, x_2^3) = x_1^3 + x_2^3$$

*A*2. If each person just consumed her endowment, $(1, 1)$, then people 1 and 2 would each have utility of 1, and person 3 would have utility of 2. So if an allocation is in the core, it must be "individually rational" : it must give people at least as high utility as they could get from their own endowments. So an allocation will be in the core only if

$$x_1^1 \geq 1 \qquad (2-1)$$

$$x_2^2 \geq 1 \qquad (2-2)$$

$$x_1^3 + x_2^3 \geq 2 \qquad (2-3)$$

Further, the allocation must be Pareto optimal if it is in the core. (Otherwise it could be blocked by a coalition of all three people.) Given that person 3 finds the 2 goods to be perfect substitutes, that person 1 likes only good 1, and that person 2 likes only good 2, an allocation $(x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_2^3)$ will be Pareto optimal (and feasible) only if

$$x_1^1 + x_1^2 + x_1^3 = 3 \qquad (2-4)$$

$$x_1^2 + x_2^2 + x_2^3 = 3 \qquad (2-5)$$

$$x_2^1 = 0 \qquad (2-6)$$

$$x_1^2 = 0 \qquad (2-7)$$

But people 1 and 2 could form a coalition of their own, and each get a utility of 2, by combining their endowments to get the allocation in which $\mathbf{x}^1 = (2, 0)$ and $\mathbf{x}^2 = (0, 2)$. So an allocation will be in the core only if it gives person 1 and person 2 each a utility of 2 or more. Given their utility functions, it must be the case then that an allocation is in the core only if

$$x_1^1 \geq 2 \qquad (2-8)$$

$$x_2^2 \geq 2 \qquad (2-9)$$

But combining $(2-8)$ and $(2-9)$ with person 3's individual rationality requirement $(2-3)$, and with the feasibility constraints $(2-4)$ and $(2-5)$, there is only one allocation left in the core : $\mathbf{x}^1 = (2,0)$, $\mathbf{x}^2 = (0,2)$ and $\mathbf{x}^3 = (1,1)$.

*Q*3. How would the equilibrium prices of the goods vary with the people's endowments in a 2–person, 2–good exchange economy, if each person's preferences could be represented by the utility function

$$u^i((\mathbf{x}^i) = -\frac{1}{x_1^i} - \frac{1}{x_2^i}$$

where x_j^i was person i's consumption of good j?

*A*3. Each person's MRS is

$$MRS^i = \frac{u_1^i}{u_2^i} = \frac{(x_2^i)^2}{(x_1^i)^2} \qquad (3-1)$$

Setting the MRS equal to the price ratio then implies that

$$\frac{(x_2^i)^2}{(x_1^i)^2} = \frac{p_1}{p_2} \qquad (3-2)$$

or

$$x_2^i = \sqrt{p_1/p_2}x_1^i \qquad (3-3)$$

Substitution of $(3-3)$ into person i's budget constraint implies that

$$p_1x_1^i + \sqrt{p_1p_2}x_1^i = m^i \qquad (3-4)$$

if m^i is person i's income, so that person i's demand function for good 1 can be written

$$x_1^i = \frac{m^i}{p_1 + \sqrt{p_1p_2}} \qquad (3-5)$$

(Since preferences here are CES, with $\rho = -1$, you can also get this demand function from the example in chapter 1 of the text.) But a person's income is the value of her endowment :

$$m^i = p_1e_1^i + p_2e_2^i \qquad (3-6)$$

so that the demand function $(3-5)$ becomes

$$x_1^i = \frac{p_1e_1^i + p_2e_2^i}{p_1 + \sqrt{p_1p_2}} \qquad (3-7)$$

The market for good 1 will clear only if total demand equals the total endowments, or

$$x_1^1 + x_1^2 = \frac{p_1e_1 + p_2e_2}{p_1 + \sqrt{p_1p_2}} = e_1 \qquad (3-8)$$

(We do not need to consider market clearing in the market for the other good, since Walras's Law guarantees that if aggregate demand for good 1 equals the aggregate endowment of good 1, then the aggregate demand for good 2 will also equal the aggregate endowment of good 2.) But equation $(3-8)$ is equivalent to

$$\frac{p_1}{p_2} = \frac{(e_2)^2}{(e_1)^2} \qquad (3-9)$$

Equation $(3-9)$ defines how the equilibrium price ratio varies with the people's endowments of the goods. Because people here have identical, homothetic preferences, the equilibrium prices depend only on the **aggregate** endowments, not on how the endowments are distributed between the 2 people.

*Q*4. Find all the pure–strategy Nash equilibria in the following strategic–form two–person game.

	LL	L	R	RR
tt	$(4,6)$	$(9,3)$	$(2,5)$	$(10,1)$
t	$(1,2)$	$(3,4)$	$(4,3)$	$(10,2)$
b	$(2,7)$	$(7,2)$	$(5,7)$	$(0,0)$
bb	$(3,5)$	$(8,6)$	$(8,8)$	$(12,3)$

*A*4. This game cannot be solved completely by iterated elimination of dominated strategies. But it can be partially solved that way.

Note first that column RR is a strictly dominated strategy for player 2. (It is dominated strictly by column L.) Once column RR is crossed out, row t of player 1 is dominated strictly by row b : so we can cross out row t.

Once row t is crossed out, column L is dominated strictly by column R for player 2, so that column L can be crossed out. With columns L and RR crossed out, row b is dominated by row bb for player 1. Crossing that row out reduces the game to the following 2–by–2 game

	LL	R
tt	$(4,6)$	$(2,5)$
bb	$(3,5)$	$(8,8)$

This new game has 2 pure–strategy Nash equilibria : (tt, LL) and (bb, R). These are the only pure–strategy equilibria to the whole game (since a strategy which has been crossed out during iterated elimination of strictly dominated strategies cannot be played in any pure–strategy (or mixed–strategy) Nash equilibrium).

There also is a mixed–strategy equilibrium to this game, in which player 1 plays tt with probability 3/4, bb with probability 1/4 and the other two strategies with probability 0, and in which player 2 plays LL with probability 6/7, R with probability 1/7, and the other 2 strategies with probability 0. That's the only mixed–strategy equilibrium. But the question did not ask about mixed–strategy equilibria.

*Q*5. Find all the Nash equilibria (in pure or mixed strategies) to the following two–person game in strategic form.

	L	R
t	$(12, 6)$	$(6, 4)$
m	$(0, 8)$	$(7, 12)$
b	$(2, 2)$	$(8, 4)$

*A*5. Note first that m is a strictly dominated (by b) strategy for player 1. That means that m will not be played with positive probability by player 1. We can restrict attention to equilibria in which the other 2 strategies are played with positive probability.

There are two pure strategies to this game, (t, L) and (b, R).

To find any mixed strategies, consider what would induce player 1 to mix between strategies t and b (since we know she will not want to play m with any positive probability). If player 2 plays L with probability β, and R with probability $1 - \beta$, then player 1 will get the expected payoffs of $12\beta + 6(1 - \beta)$ from strategy t and $2\beta + 8(1 - \beta)$ from strategy b. She will be willing to randomize only if

$$12\beta + 6(1 - \beta) = 2\beta + 8(1 - \beta)$$

or

$$\beta = \frac{1}{6}$$

Player 2 will be willing to randomize between L and R if he gets the same expected payoffs from these 2 pure strategies. Since his expected payoffs from these two pure strategies are $6\alpha + 2(1 - \alpha)$ and $4\alpha + 4(1 - \alpha)$ respectively, if player 1 plays her pure strategy t with probability α, then player 2 will be willing to randomize only if

$$6\alpha + 2(1 - \alpha) = 4\alpha + 4(1 - \alpha)$$

or

$$\alpha = \frac{1}{2}$$

So there is a mixed strategy equilibrium in which player 1's mixing probabilities over her pure strategies are $(1/2, 0, 1/2)$, and in wich player 2's mixing probabilities are $(1/6, 5/6)$.