GS/ECON 5010 Answers to Assignment 1 October 2006

*Q*1 . Are the preferences represented by the utility function below strictly monotonic? Convex?

$$u(x_1, x_2, x_3) = \sqrt{(x_1)^2 + x_2 + x_3}$$

Explain briefly.

*A*1. Recall that the preferences represented by a utility function do not change if a **monotonic increasing transformation** is applied to the utility function. So the preferences represented by $u(\mathbf{x})$ in the question are the same as the preferences represented by

$$U(\mathbf{x}) = [u(\mathbf{x})]^2 = (x_1)^2 + x_2 + x_3$$

Since $\partial U/\partial x_2 = 1 > 0$ for all x_1, and $\partial U/\partial x_3 = 1 > 0$, and $\partial U/\partial x_1 = 2x_1 \geq 0$ for all $x_1 \geq 0$, the preferences are strictly monotonic.

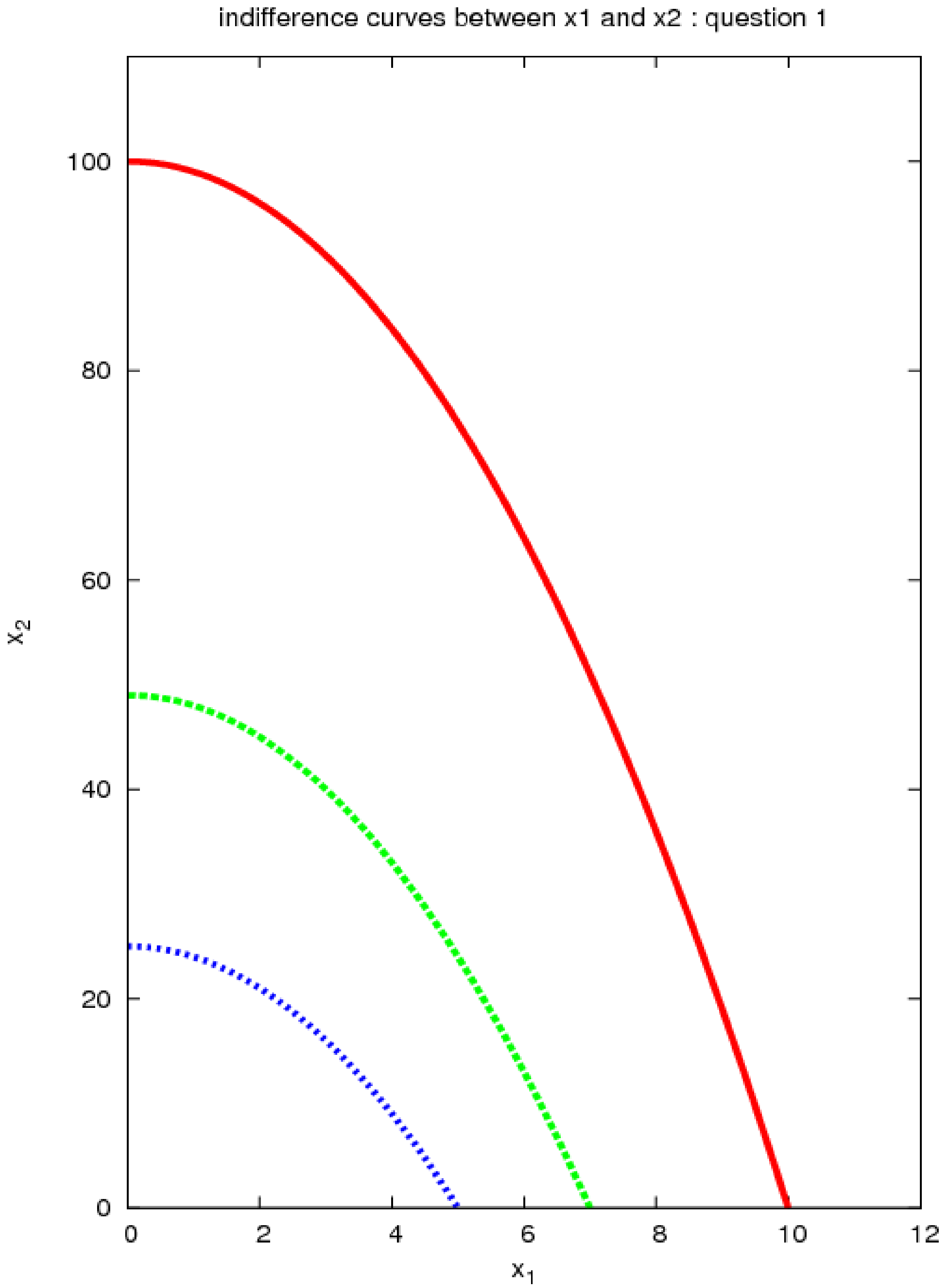
But the preferences are not convex. One way of seeing that is noting that the indifference curves between goods 1 and 2 – holding x_3 constant – have the "wrong" shape : the marginal rate of substitution $U_1/U_2 = 2x_1$ gets higher, not lower, as x_1 increases and x_2 decreases.

Another way of seeing that preferences are not convex would be to set up the 4–by–4 bordered Hessian matrix : the determinant of the top left 3–by–3 sub–matrix is negative when it should be positive.

Or, the fact that the preferences are not convex could be established more directly, by looking at the shape of the "at least as good as" sets. With these preferences, the person is indifferent between the bundles $\mathbf{x}^1 = (2, 0, 0)$ and $\mathbf{x}^2 = (0, 0, 4)$: $U(\mathbf{x}^1) = U(\mathbf{x}^2) = 4$. Now take the bundle halfway along the line connecting $\mathbf{x}^1$ and $\mathbf{x}^2$,

$$\mathbf{x}^3 = (0.5)\mathbf{x}^1 + (0.5)\mathbf{x}^2 = (1, 0, 2)$$

Since $U(\mathbf{x}^3) = 3 < 4$, $\mathbf{x}^3$ is not in the "at least as good" set corresponding to a utility level of 4 ; therefore that "as least as good as" set is not convex, so that preferences are not convex.


indifference curves between x1 and x2 : question 1
100
80
60
40
20
0
x_2
0
2
4
6
8
10
12
x_1

$Q2$. Are the preferences represented by the utility function below strictly monotonic? Convex?

$$u(x_1, x_2) = \frac{x_1 x_2}{x_1 + x_2}$$

Explain briefly. (The above definition holds only if $\mathbf{x} \neq 0$; if $\mathbf{x} = 0$, then define $u(0,0)$ here as equalling 0.)

$A2$. Note first that $u(\mathbf{x}) > 0$ whenever $\mathbf{x} \geq 0$. Second, here

$$\frac{\partial u}{\partial x_1} = \frac{1}{(x_1 + x_2)^2}[x_2(x_1 + x_2) - x_1 x_2] = (\frac{x_2}{x_1 + x_2})^2$$

and

$$\frac{\partial u}{\partial x_2} = (\frac{x_1}{x_1 + x_2})^2$$

when $\mathbf{x} \neq 0$. So both partial derivatives are non–negative, and at least one of them is strictly positive. The preferences represented by the utility function are strictly monotonic.

In this case, the determinants of the 3–by–3 bordered Hessian matrix

$$\begin{pmatrix} 0 & u_1 & u_2 \\ u_1 & u_{11} & u_{12} \\ u_2 & u_{12} & u_{22} \end{pmatrix}$$

do alternate correctly : the determinant of the 2–by–2 top left sub–matrix is negative, and that of the matrix itself is positive.

But it probably is easier simply to look at the indifference curves. Since there are only 2 goods, and since the preferences are strictly monotonic, a necessary and sufficient condition for convexity of preferences is that the indifference curves all have the "bowed out to the origin" shape. The set of bundles (x_1, x_2) which are on the same indifference curve are the bundles such that

$$\frac{x_1 x_2}{x_1 + x_2} = A \qquad (2-1)$$

for some constant A. Re–arranging equation $(2-1)$,

$$x_1 x_2 = A(x_1 + x_2) \qquad (2-2)$$

or

$$x_2 = \frac{A x_1}{x_1 - A} \qquad (2-3)$$

The slope of the indifference curve defined by equation $(2-3)$ is

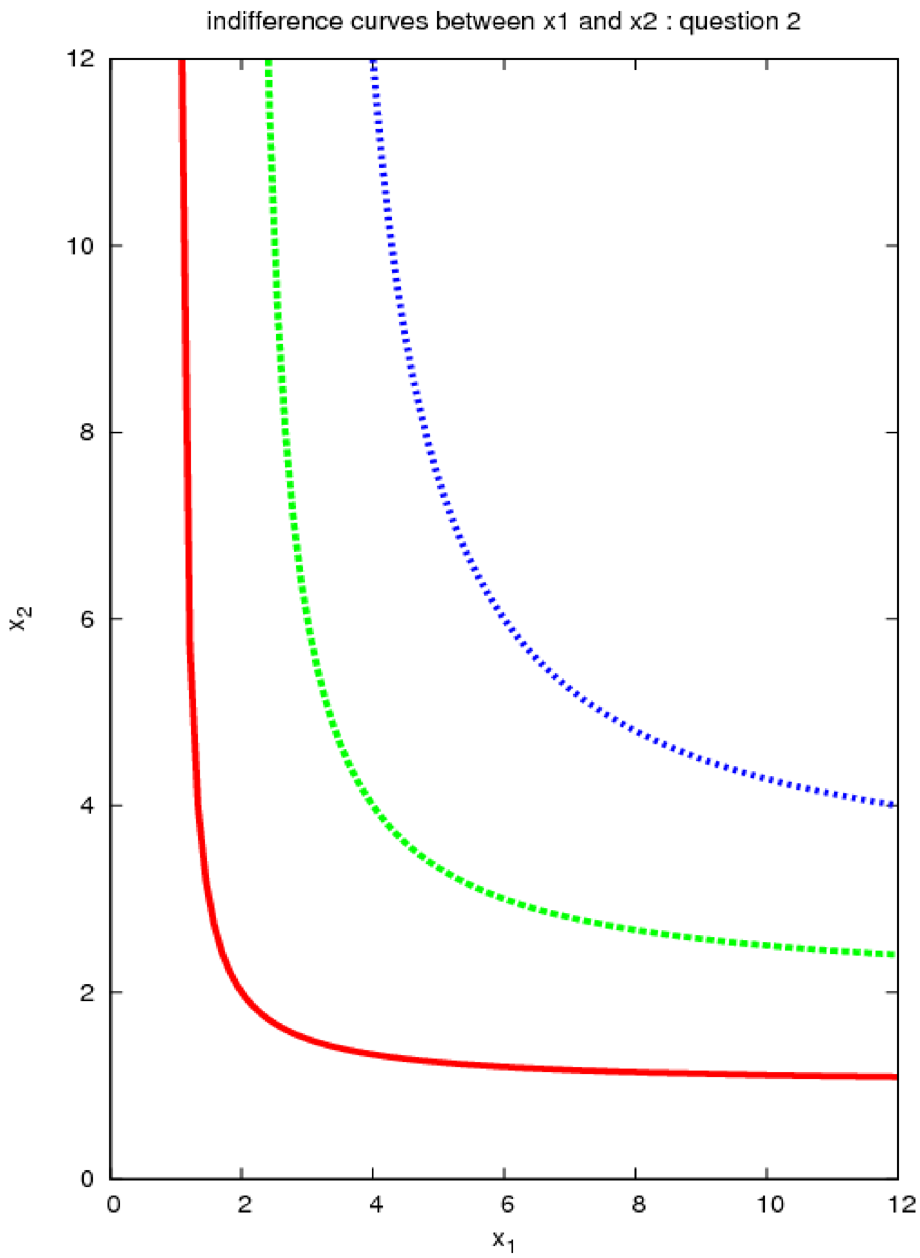
$$\frac{dx_2}{dx_1}|_{u=A} = -(\frac{A}{x_1 - A})^2 \qquad (3-4)$$

We must have $x_1 > A$ when $\mathbf{x} >> 0$; $A = u(x_1, x_2) = x_1 \frac{x_2}{x_1 + x_2} < x_1$. So as x_1 increases, $\frac{A}{x_1 - A}$ decreases ; the indifference curves all get less steep as we move down and to the right, so that preferences are convex.

Another way to establish convexity of preferences is to do a monotonically increasing transformation. If

$$U(\mathbf{x}) = \ln [u(\mathbf{x})] = \ln x_1 + \ln x_2 - \ln (x_1 + x_2)$$

then $U(\mathbf{x})$ is a concave function. The 2–by–2 matrix of its second derivatives has negative entries on the diagonal, and a positive determinant.



$Q3$. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = \ln x_1 + \ln(x_2 + x_3)$$

$A3$. The difficult part here is the fact that goods 2 and 3 are perfect substitutes for each other. Why? The marginal rate of substitution between goods 2 and 3, defined as

$$MRS_{23} \equiv \frac{u_2}{u_3}$$

equals 1 : it does not change as quantities consumed of the 2 good vary. That means that the person will never want to purchase positive quantities of both good 2 and good 3, unless $p_2 = p_3$. If she optimizes, she won't purchase the more expensive of goods 2 and 3.

For example, suppose that $p_2 > p_3$, and suppose it were true that $x_2 > 0$. Now consider lowering x_2 by 1, and raising x_3 by p_2/p_3, keeping x_1 constant. This change leaves total expenditure unchanged (since the person spends p_2 less on good 2, and $p_3(p_2/p_3)$ more on good 3). But it would raise the person's utility from $\ln x_1 + \ln(x_2 + x_3)$ to $\ln x_1 + \ln(x_2 - 1 + x_3 + p_2/p_3)$. So the original consumption plan, with $x_2 > 0$, could not have been optimal.

So the person's optimal consumption plan must have $x_2 = 0$ if $p_2 > p_3$.

Now we can proceed to solve for x_1 and x_3 (if $p_2 > p_3$). The person chooses x_1 and x_3 to maximize $\ln x_1 + \ln x_3$, subject to $p_1 x_1 + p_3 x_3 = y$. That's a standard two–good, Cobb–Douglas optimization problem, with solution $x_1 = y/(2p_1)$, $x_3 = y/(2p_3)$.

So if $p_2 > p_3$, then the consumer's Marshallian demands are

$$x_1(\mathbf{p}, y) = \frac{y}{2p_1}, x_2(\mathbf{p}, y) = 0, x_3(\mathbf{p}, y) = \frac{y}{2p_3}$$

On the other hand, if $p_3 > p_2$, then the person would not want to consume any of good 3, so that her Marshallian demand functions must be

$$x_1(\mathbf{p}, y) = \frac{y}{2p_1}, x_2(\mathbf{p}, y) = \frac{y}{2p_2}, x_3(\mathbf{p}, y) = 0$$

Finally, if it were the case that $p_2 = p_3$, then the consumer would not care what x_2 or x_3 were, just their sum ; her optimum would be any (x_1, x_2, x_3) with $x_1 = y/2p_1$, and $x_2 + x_3 = y/2p_2 = y/2p_3$.

$Q4$. Calculate a person's Marshallian demand functions, her indirect utility function, and her expenditure function, if her direct utility function is

$$u(x_1, x_2, x_3) = x_1 - \frac{1}{x_2 x_3}$$

when $y^3 > 8p_1p_2p_3$.

$A4$. Since $u_1(\mathbf{x}) = 1$, the first–order conditions for consumer optimization with respect to x_2 and x_3 can be written

$$u_2(\mathbf{x}) = \frac{1}{(x_2)^2 x_3} = \frac{p_2}{p_1} \qquad (4-1)$$

$$u_3(\mathbf{x}) = \frac{1}{x_2 (x_3)^3} = \frac{p_3}{p_1} \qquad (4-2)$$

Equations $(4-1)$ and $(4-2)$ together imply that

$$\frac{x_2}{x_3} = \frac{p_3}{p_2} \qquad (4-3)$$

Substituting for x_2 using $(4-3)$, equation $(4-2)$ becomes

$$(x_3)^3 = \frac{p_1 p_2}{(p_3)^2}$$

which can be re–written as a Marshallian demand function for good 3 :

$$x_3^M(\mathbf{p}, y) = [\frac{p_1 p_2}{(p_3)^2}]^{1/3} \qquad (4-4)$$

Similarly

$$x_2^M(\mathbf{p}, y) = [\frac{p_1 p_3}{(p_2)^2}]^{1/3} \qquad (4-5)$$

Substituting from the budget constraint $p_1x_1 + p_2x_2 + p_3x_3 = y$, the Marshallian demand function for good 1 is

$$x_1^M(\mathbf{p}, y) = \frac{y}{p_1} - 2[\frac{p_2 p_3}{(p_1)^2}]^{1/3} \qquad (4-6)$$

(Expression $(4-6)$ will be non–negative if and only if $y/p_1 \geq 2[(p_2p_3(p_1)^{-2}]^{1/3}$, which is equivalent to $y^3 > 8p_1p_2p_3$.)

To get the indirect utility function, substitute from $(4-4)$, $(4-5)$ and $(4-6)$ into the direct utility function :

$$v(\mathbf{p}, y) = \frac{y}{p_1} - 2[(p_1)^{-2}p_2p_3]^{1/3} - [(p_1)^{-2}p_2p_3]^{1/3} = \frac{y}{p_1} - 3[(p_1)^{-2}p_2p_3]^{1/3} \qquad (4-7)$$

Since $v(\mathbf{p}, e(\mathbf{p}, u)) = u$, equation $(4-7)$ implies that

$$u = \frac{e(\mathbf{p}, u)}{p_1} - 3[(p_1)^{-2}p_2p_3]^{1/3}$$

Or

$$e(\mathbf{p}, u) = p_1 u + 3[p_1 p_2 p_3]^{1/3} \tag{4 − 8}$$

[Differentiation of equation $(4-8)$ with respect to p_2 and p_3 shows that the Hicksian demand functions, $x_i^H(\mathbf{p}, u) = e_i(\mathbf{p}, u)$ are the same as the Marshallian demand functions $(i = 2, 3)$: the fact that Marshallian demand functions for goods 2 and 3 are independent of the person's income, and the Slutsky equation, imply that Hicksian and Marshallian demands for goods 2 and 3 be the same.]

*Q*5. What is a person's expenditure function, if her preferences could be represented by the utility function

$$u(x_1, x_2) = x_1 - \frac{x_1}{x_2}$$

if $p_2 < y$?

*A*5. The first–order condition for cost minimization can be written

$$\frac{u_1}{u_2} = \frac{1 - 1/x_2}{(x_1/(x_2)^2)} = \frac{p_1}{p_2} \tag{5 − 1}$$

which can also be written

$$\frac{(u/x_1)}{(x_1/(x_2)^2)} = \frac{p_1}{p_2} \tag{5 − 2}$$

or

$$[\frac{x_1}{x_2}]^2 = u\frac{p_2}{p_1} \tag{5 − 3}$$

which means that

$$\frac{x_1}{x_2} = \sqrt{u\frac{p_2}{p_1}} \tag{5 − 4}$$

The fixed utility constraint implies that $x_1 = u + (x_1/x_2)$, which, from $(5-4)$, implies that

$$x_1^H(\mathbf{p}, u) = u + \sqrt{u\frac{p_2}{p_1}} \tag{5 − 5}$$

is the Hicksian demand function for good 1.

Plugging $(5-5)$ back into $(5-4)$, therefore

$$x_2^H(\mathbf{p}, u) = 1 + \sqrt{u\frac{p_1}{p_2}} \tag{5 − 6}$$

is the Hicksian demand function for good 2.

Now the value of the expenditure function is $p_1 x_1^H(\mathbf{p}, u) + p_2 x_2^H(\mathbf{p}, u)$, or

$$e(\mathbf{p}, u) = p_1 u + \sqrt{u p_1 p_2} + p_2 + \sqrt{u p_1 p_2} = p_1 u + 2\sqrt{u p_1 p_2} + p_2 = (\sqrt{p_1 u} + \sqrt{p_2})^2 \tag{5 − 7}$$

[Note that differentiation of $(5-7)$ with respect to p_1 and p_2 yields $(5-5)$ and $(5-6)$, confirming that $x_i^H(\mathbf{p}, u) = e_i(\mathbf{p}, u)$ here.]