*Q*1. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at four different times, and the quantities x^t of the 3 goods chosen at the four different times. (For example, the second row indicates that the consumer chose the bundle $\mathbf{x} = (4, 15, 20)$ when the price vector was $\mathbf{p} = (5, 8, 1)$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	10	1	1	5	10	20
2	5	8	1	4	15	20
3	5	1	8	2	30	10
4	8	5	5	4	12	12

*A*1. The table below indicates the cost of each bundle in each period. (For example, bundle 2, $(4, 15, 20)$ cost $5 \cdot 4 + 1 \cdot 15 + 8 \cdot 20 = 195$ in period 3, so that the number 195 appears in the second column of the third row.)

80	75	60	64
125	160	260	128
195	195	120	128
190	207	216	152

A bundle $\mathbf{x}^i$ is revealed preferred to another bundle $\mathbf{x}^j$, if the person could have afforded bundle $\mathbf{x}^j$ in period i (but chose bundle $\mathbf{x}^i$ instead), that is if $\mathbf{p}^i \cdot \mathbf{x}^i \geq \mathbf{p}^i \cdot \mathbf{x}^j$. That is the same as $M_{ii} \leq M_{ij}$, where M_{ij} denotes the number in the i–th row and j-th column of the matrix above.

So the first row of the matrix indicates that bundle $\mathbf{x}^1$ is revealed preferred to all 3 of the other bundles : its cost (80) in period 1 is greater than the cost (75, 60 or 64) of any of the other three bundles. The second row indicates that bundle $\mathbf{x}^2$ is revealed preferred to bundle $\mathbf{x}^1$, since $125 < 160$, and to bundle $\mathbf{x}^4$ as well. The third and fourth row indicate nothing about what is revealed preferred, because in each case the bundle actually chosen is less expensive than any of the other bundles.

So there is only one violation of $WARP$: bundle $\mathbf{x}^1$ was chosen in period 1 when the person could have afforded $\mathbf{x}^2$, whereas bundle $\mathbf{x}^2$ was chosen in period 2 when she could have afforded $\mathbf{x}^1$. There are no other cycles of the form : bundle i revealed preferred to bundle j revealed preferred to bundle k revealed preferred to bundle i. So the violation of $WARP$ in the data is also the only violation of $SARP$.

$Q2$. The following table lists the prices of 3 goods, and the quantities a consumer chose of the goods, in 4 different years.

From these data, what can be concluded about how well off the consumer was in the different years? Explain briefly.

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	10	1	1	5	10	20
2	5	8	1	6	10	8
3	5	1	8	9	20	5
4	8	5	5	15	8	8

$A2$. Again, the matrix below lists the costs of the different bundles in diferent periods, so that element M_{ij} of the matrix below is the cost of bundle $\mathbf{x}^j$ using prices $\mathbf{p}^i$.

80	78	115	166
125	118	210	147
195	104	105	147
190	138	197	200

We can learn something about the person's preferences only if $M_{ii} \geq M_{ij}$ for some $i \neq j$: if $M_{ii} \geq M_{ij}$ then the person chose bundle $\mathbf{x}^i$ when she could have afforded bundle $\mathbf{x}^j$, so that she has shown that she prefers bundle $\mathbf{x}^i$ to bundle $\mathbf{x}^j$.

The fourth row is the most informative. Bundle $\mathbf{x}^4$ cost more in period 4 than any of the other three bundles. So her behaviour in period 4 shows that she prefers bundle $\mathbf{x}^4$ to any of the other 3 bundles.

Turning to the other periods, in period 1, she actually had 80 dollars to spend (the cost of the bundle she actually chose, bundle $\mathbf{x}^1$). The only other bundle which cost 80 dollars or less in period 1 is bundle $\mathbf{x}^2$, which cost 78 dollars. So her behaviour in period 1 shows that she prefers bundle $\mathbf{x}^1$ to bundle $\mathbf{x}^2$.

In period 2, she had 118 dollars to spend, and all of the other bundles (other than the one she chose, $\mathbf{x}^2$) cost more than 118, so we learn nothing about her preferences from her behaviour in period 2. In period 3 she had 105 dollars to spend : since bundle $\mathbf{x}^2$ would have cost only 104 dollars in period 3, her period–3 behaviour reveals that she prefers bundle $\mathbf{x}^3$ to bundle $\mathbf{x}^2$.

So there are no violations of $WARP$ or $SARP$ in this consumer's choices. From her choices, she has revealed that she likes bundle $\mathbf{x}^4$ better than the other 3 bundles, and likes bundle $\mathbf{x}^2$ less than the other bundles. The only think that her choices do not reveal is whether she likes bundle $\mathbf{x}^1$ more than $\mathbf{x}^3$. Her rankings of the 4 bundles could be $\mathbf{x}^4 \succ \mathbf{x}^3 \succ \mathbf{x}^1 \succ \mathbf{x}^2$, or $\mathbf{x}^4 \succ \mathbf{x}^3 \sim \mathbf{x}^1 \succ \mathbf{x}^2$ or $\mathbf{x}^4 \succ \mathbf{x}^1 \succ \mathbf{x}^3 \succ \mathbf{x}^2$.

$Q3$. A country contains thousands of identical firms, each of which have initial wealth of 4, and each of which is run by an identical risk–averse entrepreneur, with a utility–of–wealth function

$$u(W) = \sqrt{W}$$

Each entrepreneur faces a choice between 2 projects. Project s offers a sure gain of 4 (on top of the entrepreneur's initial wealth of 4). Project r offers a chance at a gain of G, with probability 0.5, but will cause the entrepreneur to lose everything (including her initial wealth of 4) with probability 0.5.

The outcome of any individual entrepreneur's risky project r is independent of the outcome of any other entrepreneur's risky project.

i For what values of G would an entrepreneur prefer to invest in project r?

ii If the entrepreneurs each owned an equal (small) share of each of the firms, for which values of G would they prefer to invest in project r?

$A3$. *i* The expected utility from project s is $\sqrt{8}$. The expected utility from project r is $(0.5)\sqrt{4+G}$. Therefore she will prefer the risky project if and only if

$$\sqrt{8} < (0.5)\sqrt{4+G} \tag{3-1}$$

Squaring both sides of $(3-1)$, it is equivalent to

$$8 < \frac{1}{4}(4+G) \tag{3-2}$$

or

$$G > 28$$

ii Since the returns to each entrepreneur's project are independent, sharing each others' returns enables the entrepreneurs to pool their risks. The law of large numbers says that the return to owning a share $1/N$ of the returns to each of N risky projects approaches the expected return to any of the projects, as $N \to \infty$. That expected return is $(G-4)/2$, since each project produces a net gain of G with probability 0.5 and a net loss of 4 with probability (0.5).

Assuming that 1000 entrepreneurs is a large enough number that the law of large numbers is valid, then each entrepreneur will prefer her share in the risky projects if that expected return $(G-4)/2$ exceeds the net return 4 of the safe project. So the values of G which make the risky project more attractive are those for which

$$(G-4)/2 > 4 \tag{3-3}$$

or $G > 12$

*Q*4. Suppose that $G = 9$ in the model of question 3 above. Suppose that a government programme to encourage entrepreneurial activity is introduced, in which losses from the risky project r are covered completely. Entrepreneurs are guaranteed that their wealth stays at its initial level of 4, even if they invest in project r and get a bad outcome.

This project is funded by a tax T on all successful entrepreneurs : entrepreneurs who invest in project s, and those who invest in project r and get the good outcome each must pay a tax of T.

For what values of T will entrepreneurs decide to undertake project r, rather than s?

*A*4. Now the entrepreneur's expected utility if she invests in the safe project is

$$\sqrt{8-T}$$

and her expected utility if she invests in the risky project is

$$(0.5)\sqrt{4+G-T} + (0.5)\sqrt{4} = (0.5)\sqrt{13-T} + (0.5)\sqrt{4}$$

so that she will prefer the risky project if

$$(0.5)\sqrt{13-T} + (0.5)\sqrt{4} > \sqrt{8-T} \qquad (4-1)$$

which is equivalent to

$$\sqrt{13-T} + 2 > 2\sqrt{8-T} \qquad (4-2)$$

Squaring both sides of $(4-2)$, it is equivalent to

$$(13-T) + 4\sqrt{13-T} + 4 > 4(8-T) \qquad (4-3)$$

or

$$4\sqrt{13-T} > 15 - 3T \qquad (4-4)$$

which, in turn (squaring both sides again) is equivalent to

$$16(13-T) > 225 - 90T + 9T^2 \qquad (4-5)$$

which is a quadratic expression which can be written

$$9T^2 - 74T + 17 < 0 \qquad (4-6)$$

There are two roots to the quadratic function on the left side of expression $(4-5)$, 0.23653 and 7.98568. However, $T = 7.98568$ does not solve equation $(4-1)$ with equality. (The problem? It was squaring both sides of $(4-4)$: at $T = 7.98568$ the left side of $(4-4)$ is positive, the right side is negative, but the two sides are equal in absolute value.)

You can check that $EU(r) - EU(s)$ is a monotonically increasing function of T. So it will be positive whenever $T > 0.23653$.

The entrepreneurs will prefer the risky project if the tax T levied on all successful entrepreneurs is greater than 0.23653

*Q*5. Now suppose that the government entrepreneurial insurance programme from question #4 must break even : the revenues collected from the tax must exactly cover the compensation for losses from the risky project.

Each entrepreneur takes the tax payable T as given, and makes her own expected–utility–maximizing choice. An equilibrium tax T is one which leads to the insurance programme breaking even, given this behaviour.

Find two equilibrium values for T.

*A*5. Suppose that no entrepreneurs chose the risky project. Then there would be no losses to insure. In this case $T = 0$. But if $T = 0$, then each entrepreneur would prefer the safe project (even though there is — free — insurance provided against loss in the risky project). When $T = 0$, the expected utility from the safe project is $\sqrt{8}$ and the return from the risky project is $(0.5)\sqrt{13} + (0.5)\sqrt{4}$, and it can be checked that $\sqrt{8} > (0.5)\sqrt{13} + (0.5)\sqrt{4}$. (The answer to question 4 above shows that, at $T = 0$, the safe project offers higher expected utility than the risky project.)

So there is an equilibrium in which no tax is levied, and no insurance is needed, since there are no risky projects to insure.

Now suppose that all the entrepreneurs chose the risky project. In that case, the government would need a tax of $T = 4$ on successful entrepreneurs, since half the entrpreneurs will be compensated 4 dollars each for their losses (paid for by taxes on the other half). If $T = 4$,what project would entrpreneurs prefer? The answer to question 4 above shows that they would prefer the risky project. Or it can be checked directly : the safe project gives them a sure wealth of $8 - T = 4$; the risky project gives them wealth of 4 with probability 0.5 (if they fail) and $13 - T = 9$ with probability 0.5, which is certainly the more attractive option.

So there is also an equilibrium in which all successful entrpreneurs pay a tax of 4, and in which all entrpreneurs prefer the risky project. (Note that this second equilibrium is worse for everyone than the first equilibrium, since the generous insurance programme encourages excessive risk–taking.)

In fact, there is a third equilibrium. If (approximately) 11.166 percent of the entrepreneurs choose the risky project, then the required tax will be 0.23653 : the total expected payments to unsuccessful entrepreneurs would be $(0.11166)(0.5)(4)$, which would equal the expected tax revenue $(0.23653)(1 - (0.11166)/2)$ on the remaining successful entrepreneurs. In this case, each entrepreneur would not care whether to undertake the risky project or the safe project, since both offer the same expected utility when $T = 0.23653$.