GS/ECON 5010 Answers to Assignment 3 November 2006

$Q1$. What is the equation of the supply curve of a firm which has a total cost function with the equation

$$TC(q) = (q-4)^3 + 10q + 64$$

where q is the quantity of output produced by the firm?

$A1$. Given the total cost function, the firm's marginal cost function is the derivative of TC with respect to q, or

$$MC(q) = 3(q-4)^2 + 10$$

Note that the marginal cost curve is U–shaped, decreasing for $q < 4$ and increasing for $q > 4$.

The average cost is TC divided by q, or

$$AC(q) = 10 + \frac{3(q-4)^2 + 64}{q}$$

Since

$$(q-4)^3 = q^3 - 12q^2 + 48q - 64$$

therefore

$$AC(q) = q^2 - 12q + 58$$

To find the shape of the average cost curve, differentiate AC with respect to q :

$$AC'(q) = 2q - 12$$

The average cost curve is U–shaped, with a minimum at $q = 6$. As must be the case $AC(q) = MC(q) = 22$ at $q = 6$.

The firm's supply curve is the upward–sloping part of the marginal cost curve, above its intersection with the average cost curve.

The minimum average cost is 22. So if $p < 22$ the firm chooses not to produce any of the good. If $p > 22$, the firm's quantity supplied q is determined by $MC(q) = p$, or $3(q-4)^2 + 10 = p$, implying that

$$q = 4 + \sqrt{(p-10)/3}$$

if $p > 22$.

$Q2$. What is the equation of the long–run supply curve for an industry, if the industry contained 100 firms, each with a (long–run) total cost function $TC(q) = (q-4)^3 + 10q + 64$, 100 more firms, each with a (long–run) total cost function $TC(q) = (q-4)^3 + 20q + 64$, and 100 more firms, each with a (long–run) total cost function $TC(q) = (q-4)^3 + 30q + 64$?

$A2$. The answer to question #1 can be generalized to yield the supply curve for a firm for which the total cost function is

$$TC(q) = (q-4)^3 + Aq + 64$$

for any $A > 0$. Such a firm has a U–shaped average cost curve, with a minimum average cost of $A + 12$ at $q = 6$. So it supplies nothing if the price is less than $A + 12$, and supplies a quantity

$$q = 4 + \sqrt{(p-A)/3}$$

if $p > A + 12$.

In question #2, there are 100 firms for which $A = 10$, 100 for which $A = 20$ and 100 for which $A = 30$.

So if $p < 22$, none of the firms will be willing to supply anything. If $22 < p < 32$, then the 100 firms for which $A = 10$ are all willing to supply positive quantities of the good, but not the other two types of firm. In this case, the industry supply curve has the equation

$$Q = 400 + 100\sqrt{(p-10)/3}$$

If $32 < p < 42$, then firms for which $A = 10$ and $A = 20$ will be willing to supply positive quantities of the good, but not firms for which $A = 30$. The aggregate quantity supplied is

$$Q = 800 + 100\sqrt{(p-10)/3} + 100\sqrt{(p-20)/3}$$

Finally, if $p > 42$, then all 300 firms are willing to supply positive quantities of the good. The quantity supplied is

$$Q = 1200 + 100\sqrt{(p-10)/3} + 100\sqrt{(p-20)/3} + 100\sqrt{(p-30)/3}$$

(Note that the quantity supplied here is discontinuous at $p = 22$, $p = 32$ and $p = 42$.)

$Q3$. Suppose that the aggregate quantity demanded of a product by all males in a market had the equation

$$Q^m = 24 - \frac{p^m}{2}$$

where Q^m is the quantity demanded by men, and p^m the price charged to men, and that the aggregate quantity demanded by all females was

$$Q^f = 24 - p^f$$

(where Q^f is the quantity demanded by females, and p^f the price charged to females)?

The product is supplied by a monopoly, with a constant marginal cost of production of 4.

What profit would the monopoly earn if it could charge different prices to males and females? What profit would it earn if it had to charge to same price to males as to females?

*A*4. Start with the market among males, if they could be charged a separate price. Since

$$Q^m = 24 - \frac{p^m}{2}$$

then the monopoly's inverse demand function is

$$p^m = 48 - 2Q^m$$

With a straight line demand curve, the marginal revenue curve starts out at the same level as price when $Q = 0$, and has twice as large a slope, so that

$$MR^m = 48 - 4Q^m$$

A profit–maximizing monopoly finds the quantity Q^m for which $MR = MC$, or

$$48 - 4Q^m = 4$$

so that

$$Q^m = 11$$

[Alternatively, this result can be obtained directly by maximizing $p^m(Q^m) - 4Q^m = (48 - 4Q^m)Q^m - 4Q^m$ with respect to Q^m.]

To sell 11 units of the good, the monopoly must charge males a price of $p^m = 48 - 2(11) = 26$. So it sells 11 units at a price of 26 each, for a total profit of $(p^m - MC)Q^m = (26 - 4)(11) = 242$.

In the market among females, $Q^f = 24 - p^f$, so that

$$p^f = 24 - Q^f$$

implying a marginal revenue curve of

$$MR^f = 24 - 2Q^f$$

The profit–mazimizing quantity to sell to females, for which $MR^f = MC$ is the solution to $24 - 2Q^f = 4$, or

$$Q^f = 10$$

so that $p^f = 14$, and profits on sales to females are $(p^f - MC)Q^f = (14 - 4)(10) = 100$.

If the monopoly can price discriminate by charging a higher price to males than to females, then its total profits are $100 + 242 = 342$.

If the monopoly must charge the same price p to all buyers, then it faces an aggregate demand curve of

$$Q = Q^f + Q^m = (24 - p) + (24 - p/2) = 48 - 3p/2$$

[This equation defines the aggregate demand only if $p \leq 24$: if $p > 24$ then males buy a quantity of 0.]

The inverse demand function, if it must charge the same price to all, is

$$p = 32 - \frac{2}{3}Q$$

which implies a marginal revenue function of

$$MR = 32 - \frac{4}{3}Q$$

Setting $MR = MC$ means choosing a Q such that $32 - \frac{4}{3}Q = 4$, or

$$Q = 21$$

The monopoly can sell 21 units only if it charges a price of $32 - \frac{2}{3}21 = 18$. [At this price, it sells 15 units to men and 6 units to women.]

Its profits are $(p - MC)Q = (18 - 4)(21) = 294$. Even though here the monopoly sells the same total quantity, 21, whether or not it can price discriminate between the sexes, its profits are higher when it can practise price discrimination.

*Q*4. Find a Cournot equilibrium for an industry containing 10 identical firms, each of which had a total cost function

$$TC(q) = 15 + q \quad q > 0$$
$$= 0 \quad q = 0$$

if the market demand for the good produced by the firms was

$$Q = 13 - p$$

where p was the price of the good?

*A*4. From section 4.2.1 of *Jehle and Reny*, if there are J ientical firms in a Cournot oligopoly, each with identical constant marginal costs c, each firm's profit will be

$$\pi_j = \frac{(a-c)^2}{(J+1)^2 b}$$

if the demand curve for the oligopoly's output has the equation

$$Q = a - bp$$

In this question, $c = 1$, $a = 13$ and $b = 1$, so that the formula implies profits per firm of 144/121 if all 10 firms produced their equilibrium Cournot outputs $(a - c)/((J + 1)b) = 12/11$.

But each firm has fixed costs of 15 here. If all 10 firms produce positive levels of output, then they will each earn negative profits in equilibrium. This outcome then cannot be an equilibrium : firms can always ensure 0 profits by choosing an output level of 0.

So in equilibrium it must be the case that some firms choose to produce nothing. There are too many firms for each of them to make a profit if they all behave non–cooperatively.

How many firms can choose positive levels of production? The largest value of J for which $\pi_j \geq 15$ here is $J = 2$. (For $J = 3$, $\pi_j = 144/16 = 9 < 15$.)

So one possible equilibrium might be for 2 of the 10 firms each to choose the two–firm Cournot output levels $(a - c)/3b$, and for each of the other 8 firms to produce nothing. Here, this would imply output levels of 4 for each of the two firms which produce positive levels of output, yielding a price of 5, and profits of $(5)(4) - (4 + 15) = 1$.

Is this a Cournot–Nash equilibrium? If $q_1 = 4$, and $q_3 = q_4 = \cdots = q_{10} = 0$, then firm 2's best response is to produce $q_2 = 4$. Similarly, $q_1 = 4$ is firm 1's best response to $q_2 = 4$, $q_3 = q_4 = \cdots = q_{10} = 0$. What about firm 3? If $q_1 = q_2 = 4$ (and $q_4 = q_5 = \cdots = q_{10} = 0$), then firm 3 would earn profits of

$$(13 - 8 - q_3)q_3 - (q_3 + 15)$$

if it chose an output level of $q_3 > 0$. This expression is maximized at $q_3 = 2$, which would imply profits of $(13 - 8 - 2)2 - (2 + 15) = -11 < 0$. So firm 3's best response is to produce $q_3 = 0$, when $q_1 = q_2 = 4$, and $q_4 = q_5 = \cdots = q_{10}$. It is a Cournot equilibrium if two firms each produce 4,and the other 8 each produce 0.

[Note that it actually is not sufficient merely to notice that profits in a J–firm Cournot equilibrium are negative if $J \geq 3$. That would make sense (from the prospective of a third firm) only if firms 1 and 2 were producing at 3–firm–Cournot levels of output. They're not : they're producing at 2–firm–Cournot levels. It must be shown that a third firm cannot produce profitably, if the first two firms were producing at the 2–firm Cournot equilibrium levels.]

Could there be any other equilibria? The only possible equilibrium if J firms produce positive quantities of output is the symmetric J–firm Cournot equilibrium. This cannot yield non–negative profits if $J > 2$.

So the remaining case to test is one in which only one firm produces positive quantities. A monopoly would choose an output level of $(a-c)/2b = 6$, and earn profits of $(13-6)(6)-(6+15) = 21 > 0$. Would the other firms be willing to produce $q_i = 0$ when $q_1 = 6$? Firm 2's profit would be $(13 - 6 - q_2)q_2 - (q_2 + 15)$ if firm #1 chose $q_1 = 6$, and if $q_2 > 0$. This profit is a concave function of q_2, maximized at $q_2 = 3$. In this case, firm 2 would earn $(13 - 6 - 3)3 - (3 + 15) = -6 < 0$, and so would be better off not producing at all.

So there are two types of equilibrium in this industry. There can be an equilibrium in which 2 firms each produce 4 units, and the other 8 all produce nothing. Or there can be an equilibrium in which 1 firm produces 6, and in which the other 9 firms each produce nothing.

*Q*5. Suppose that the firms in the industry described in question #4 above chose prices simultaneously, instead of quantities. Just as in the Bertrand model, consumers all buy from the lowest–priced firm, with the following modification to the Bertrand model : if all firms charge the same price, then all the consumers choose to buy from firm #1 (and if two firms i and j charged the lowest price, with $i < j$, all consumers would buy from firm $\#i$, and so on).

What would the equilibrium be in this industry?

*A*5. To start, consider the profits of a single firm, as a function of its price p, if it were a monopoly. It would sell $(13 - p)$ units, and earn

$$\pi = (13 - p)p - (13 - p + 15)$$

Differentiating this expression with respect to p

$$\pi'(p) = 14 - 2p$$

So the monopoly price is 7 (resulting in a quantity sold of 6, as derived in the answer to question #4), and a profit of $(13 - 7)7 - (13 - 7 + 15) = 21$. So one firm charging a price of 7 can make positive profits, if all the consumers decided to buy from it.

These monopoly profits are also a concave function of price. They are positive whenever $(13 - p)p - (13 - p + 15) > 0$. They would equal zero whenever $(13 - p)p - (13 - p + 15) = 0$, or whenever

$$p^2 - 14p + 28 = 0 \qquad (5 - 1)$$

Solving $(5-1)$, monopoly profits will be non–negative whenever p is in the interval $[p_m, p_M]$, where

$$p_m = 7 - \sqrt{21}$$

$$p_M = 7 + \sqrt{21}$$

Now suppose that the lowest of the 10 firms' prices were strictly within the interval (p_m, p_M). (That is, suppose the lowest price was greater than p_m but less than p_M.) Let firm i be the firm choosing the lowest price. (If more than one firm charges the lowest price, then let firm i be the lowest–numbered firm charging the lowest price.) Under the assumptions of the question, every firm other than firm i would have zero sales. So if p_i was in (p_m, p_M), then some other firm j could undercut firm i slightly, capture the whole market, and earn positive profits.

If the lowest price were p_M or greater, then some other firm could choose to charge the monopoly price $7 < p_M$, undercut (by a lot) all the other firms, and make positive profits.

If the lowest price were less than p_m, then firm i would lose money, and would be better choosing a very very high price and making 0 profits.

So in Bertrand equilibrium, the lowest price must be p_m.

Suppose that the second lowest price were strictly larger than p_m, say $p_m + \epsilon$. Then firm i could raise its price to $p_m + \epsilon/2$: it would still be the lowest–priced firm, and it would now make positive profits. So, in equilibrium, there must be at least 2 firms tied for lowest.

But any situation in which 2 or more firms charge prices of p_m, and all the others charge higher prices, will be a Bertrand equilibrium here. Firm i (the lowest–numbered firm charging a price of p_m) cannot raise its price, without losing all its customers. Any other firm can attract customers only by charging a price below p_m, which would result in negative profits. So no firm can increase its profits by changing its price, so that the situation is a Bertrand equilibrium.