GS/ECON 5010 Answers to Assignment 4 November 2006

*Q*1. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of 60 units of good 1 and 120 units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = \ln x_1^1 + 2 \ln x_2^1$$

$$u^2(x_1^2, x_2^2) = 112 - \frac{1}{x_1^2} + \ln x_2^2$$

where x_j^i is person i's consumption of good j?

*A*1. The conditions for an allocation $(x_1^1, x_1^2, x_2^1, x_2^2)$ to be on the contract curve are that

i it be feasible : $x_1^1 + x_1^2 = 60$, $x_2^1 + x_2^2 = 120$

ii it be efficient : if each x_j^i is positive, then $MRS^1(x_1^1, x_2^1) = MRS^2(x_1^2, x_2^2)$

In writing condition i as an equality, I have used the fact that both people's preferences are strictly monotonic, so that it is inefficient to waste any of either good. Since each person's consumption of each good must be non–negative, it must also be the case that

iii $0 \leq x_1^1 \leq 60$, $0 \leq x_2^1 \leq 120$

Given the preferences

$$MRS^1(x_1^1, x_2^1) \equiv \frac{u_1^1(x_1^1, x_2^1)}{u_2^1(x_1^1, x_2^1)} = \frac{x_2^1}{2x_1^1}$$

$$MRS^2(x_2^1, x_2^2) \equiv \frac{u_1^2(x_1^2, x_2^2)}{u_2^2(x_1^2, x_2^2)} = \frac{x_2^2}{(x_1^2)^2}$$

so that condition ii is

$$\frac{x_2^1}{2x_1^1} = \frac{x_2^2}{(x_1^2)^2} \tag{1-1}$$

Substituting from the feasibility constraints i for x_1^2 and x_2^2, equation $(1-1)$ becomes

$$\frac{x_2^1}{2x_1^1} = \frac{(120 - x_2^1)}{(60 - x_1^1)^2} \tag{1-2}$$

which can be re–arranged into

$$x_2^1(60 - x_1^1)^2 = 2(120 - x_2^1)x_1^1$$

or

$$x_2^1 = \frac{240x_1^1}{(60 - x_1^1)^2 + 2x_1^1} \tag{1-3}$$

Equation $(1-3)$ defines a curve in x_1^1–x_2^1 space. This curve, for $0 \leq x_1^1 \leq 60$ is the contract curve in the Edgeworth box. Any allocation $(x_1^1, x_1^2, x_2^1, x_2^2)$ which obeys $(1-3)$, with $0 \leq x_1^1 \leq 60$, and with $x_1^2 = 60 - x_1^1$, $x_2^2 = 120 - x_2^1$ will be Pareto optimal.

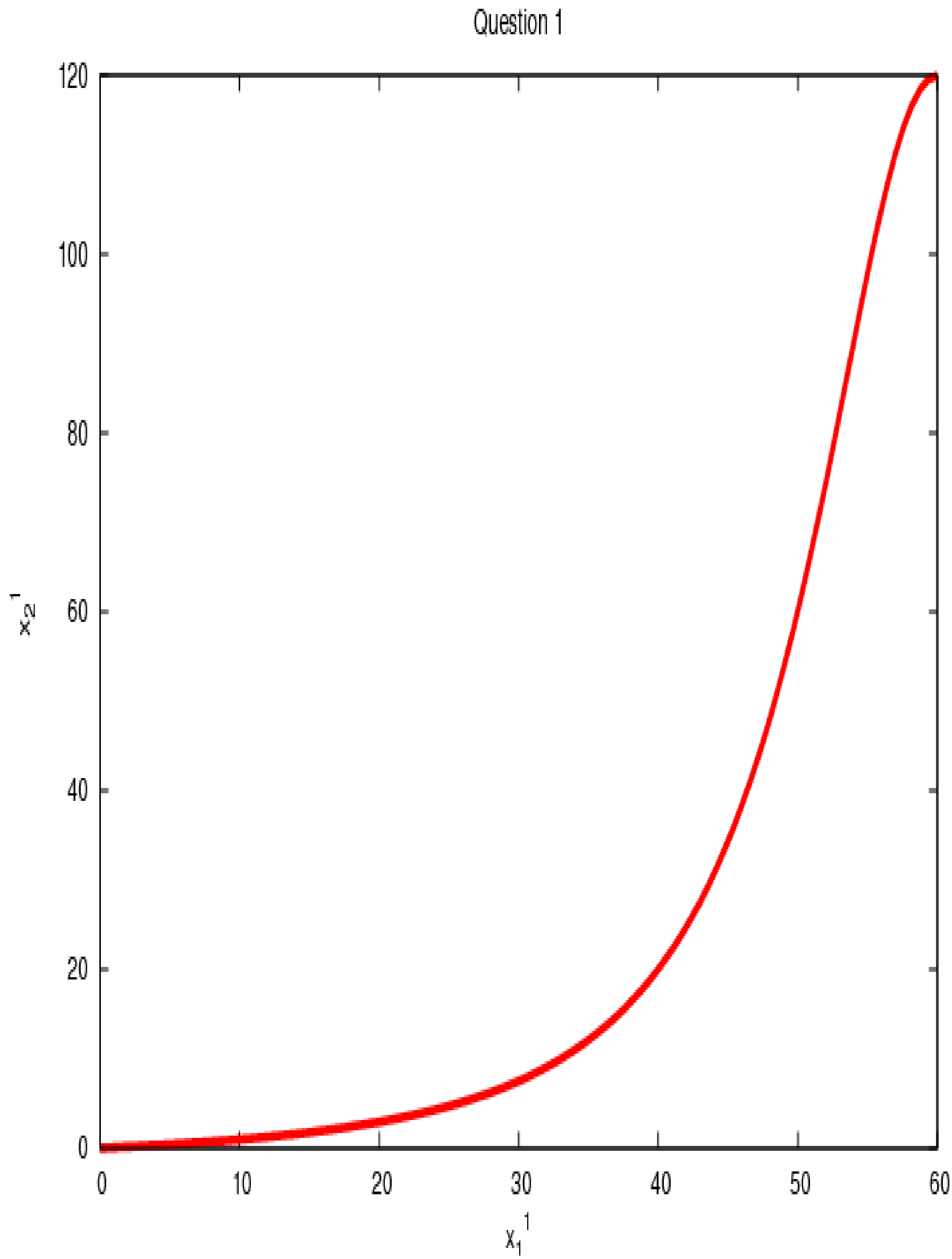

Question 1
120
100
80
60
40
20
0
0
10
20
30
40
50
60
x_2^1
x_1^1

Figure 1 illustrates the curve. It goes through the bottom left and top right corners of the Edgeworth box, and it slopes up.

The slope of the contract curve, from differentiation of $(1-3)$, is

$$\frac{240(60-x_1^1)(60+x_1^1)}{[(60-x_1^1)^2+2x]^2}$$

This slope is positive whenever $0 < x_1^1 < 60$; it equals 0 at the top right corner (where $x_1^1 = 60$) ; it equals 1/3600 at the bottom left corner (where $x_1^1 = 0$), and is "S–shaped" as in the figure.

*Q*2. What are all the allocations in the core of a 3–person, 2–good economy, in which each person's preferences can be represented by the utility function

$$u^i(x_i^1, x_2^i) = x_i^1 x_2^i$$

where x_j^i is person i's consumption of good j, and where the endowments e^i of the three people are $e^1 = (2,0)$, $e^2 = (0,2)$, $e^3 = (1,1)$?

*A*2. Here the aggregate endowment vector (E_1, E_2) is $(3,3)$. Each person's MRS is x_2^i/x_1^i. Therefore Pareto optimality requires that $x_2^1/x_1^1 = x_2^2/x_1^2 = x_2^3/x_1^3$, and that $x_1^1 + x_1^2 + x_1^3 = 3 = x_2^1 + x_2^2 + x_2^3$, so that an allocation is Pareto optimal only if

$$x_1^i = x_2^i \quad i = 1,2,3 \qquad (2-1)$$

People 1 and 2 get utility of 0 from their endowment vectors, and person 3 gets utility 1. Therefore person 3 will like the allocation (x_1^3, x_2^3) as much as her endowment if and only if $x_1^3 x_2^3 = 1$. That means that an allocation which satisfies the Pareto optimality condition $(2-1)$ will be "individually rational" (that is, at least as good as the endowment for each person) if and only if

$$x_1^3 = x_2^3 \geq 1 \qquad (2-2)$$

So any allocation $(x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_2^3)$ for which $x_1^i = x_2^i$ for each i, and for which $x_1^3 = x_2^3 \geq 1$ cannot be blocked by a coalition of all 3 people, or by any coalition of 1 person.

To be in the core, an allocation must satisfy $(2-1)$ and $(2-2)$. But not every allocation which satisfies those conditions in the core : an allocation which can be blocked by a coalition of 2 people cannot be in the core.

First of all, note that a core allocation cannot give person 3 more than 1 unit of each good. For suppose that the allocation were $\mathbf{x}^1 = (a,a), \mathbf{x}^2 = (b,b), \mathbf{x}^3 = (3-a-b, 3-a-b)$ with $a+b < 2$. Then people 1 and 2 could form a coalition of their own, and propose the allocation $\mathbf{y}^1 = (a,a), \mathbf{y}^2 = (2-a, 2-a)$ which is feasible, at least as good for both person 1 and person 2, and strictly better for person 2 if $a+b < 2$. In other words the coalition $\{1,2\}$ can block

$\mathbf{x}^1 = (a,a), \mathbf{x}^2 = (b,b), \mathbf{x}^3 = (3-a-b, 3-a-b)$ with $\mathbf{y}^1 = (a,a), \mathbf{y}^2 = (2-a, 2-a)$ whenever $a+b<2$. Therefore, if an allocation $(x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_2^3)$ is in the core, it must satisfy conditions $(2-1)$ and $(2-2')$:

$$x_1^3 = x_2^3 = 1 \qquad (2-2')$$

Now suppose the allocation gives person 1 the consumption bundle (a,a), with $a<1$, so that person 2 gets $x^2 = (2-a, 2-a) >> (1,1)$. Here person 1 is doing worse than person 2. Can she do anything about this apparent inequity?

Person 1 could attempt to form a coalition with person 3. That coalition would have a total endowment $(3,1)$ to divide between them. If 1 and 3 were to divide this endowment efficiently, then person 3 would get $(3c, c)$ for some $1 \geq c \geq 0$, and person 1 would get $(3(1-c), 1-c)$.

Would person 3 be willing to join this coalition? Only if it gave her her reservation level of utility, 1. So in order to get person 3 to join this coalition, c would have to be large enough so that $c(3c) \geq 1$, or

$$c \geq \sqrt{\frac{1}{3}} \qquad (2-3)$$

What would person 1 have left, if c satisfied $(2-3)$ with equality? She'd have the allocation

$$y^1 = (3(1-\sqrt{\frac{1}{3}}), (1-\sqrt{\frac{1}{3}})) \qquad (2-4)$$

which would give her the utility level

$$3(1-\sqrt{\frac{1}{3}})^2$$

The original allocation (which she is trying to block) gave her (a,a), with utility a^2. So her proposed coalition with person 3 can block the original allocation if and only if

$$3(1-\sqrt{\frac{1}{3}})^2 > a^2$$

or

$$\sqrt{3}(1-\sqrt{\frac{1}{3}}) > a \qquad (2-5)$$

which is the same thing as

$$\sqrt{3}-1 > a \qquad (2-6)$$

If a is so small that condition $(2-6)$ holds, then the allocation is **not** in the core : person 1 can form a coalition with person 3 so as to block the original allocation.

On the other hand, if a is large enough so that $(2-6)$ does not hold, she cannot block the allocation : she can't offer person 3 enough to join her in a blocking coalition, and she certainly can't make a deal with person 2 : if she gets more than (a,a) then person 2 would get less than $(2-a, 2-a)$.

By the same token, if person 2 got the allocation (b,b), with $b<1$, he could form a coalition with person 3 so as to block the allocation if and only if $b < \sqrt{3}-1$.

Therefore, an allocation will be in the core if and only if it satisfies $(2-1)$ and $(2-2')$, and condition $(2-7)$:

$$\mathbf{x_j^i} \geq \sqrt{3}-1 \quad i=1,2 \quad j=1,2 \tag{2-7}$$

(The actual value of $\sqrt{3}-1$ is about 0.73205, so allocations such as $(1.2.1.2, 0.8, 0.8, 1, 1)$, and $(0.9, 0.9, 1.1, 1.1, 1, 1)$ are in the core.)

*Q*3. Calculate the competitive equilibrium for the 2–person, 2–good economy described in question #1, if person 1's endowment was $(0, 30)$, and person 2's endowment was $(60, 90)$.

*A*3. Only relative prices matter in competitive economies : if (p_1, p_2) is an equilibrium price vector, then so is (kp_1, kp_2), for any $k > 0$. So, for example, we can choose one of the goods as numéraire, and set its price equal to 1.

I will let good 1 be the numéraire, so that I am looking for a price vector $(1, p)$ which makes excess demand for each good equal zero.

Person 1's income m^1 is $30p$, if her endowment is $0, 30)$, and if the price of good 2 is p. Person 2's income is $60 + 90p$ if his endowment is $(60, 90)$, if the price of good 1 is 1, and the price of good 2 is p.

Person 1 has Cobb–Douglas preferences. Her demands for goods 1 and 2 are

$$x_1^1 = \frac{m^1}{3} \tag{3-1}$$

$$x_2^1 = \frac{2m^1}{3p} \tag{3-2}$$

when the price vector is $(1, p)$. Given her endowent $(0, 30)$, $(3-1)$ and $(3-2)$ imply that

$$x_1^1 = 10p$$

$$x_2^1 = 20$$

so that her excess demands $(z_j^i \equiv x_j^i - e_j^i)$ are

$$z_1^1 = 10p \tag{3-3}$$

$$z_2^1 = -10 \tag{3-4}$$

It's trickier to calculate person 2's excess demand functions. Given his marginal rate of substitution, calculated in question #1, an allocation (x_1^2, x_2^2) will be otimal for him only if $MRS^2 = \frac{1}{p}$ or

$$x_2^2 = \frac{(x_1^2)^2}{p} \tag{3-5}$$

which can be written

$$px_2^2 = (x_1^2)^2 \qquad (3-6)$$

Plugging $(3-6)$ into person 2's budget constraint

$$x_1^2 + px_2^2 = m^2 = 60 + 90p \qquad (3-7)$$

yields

$$x_1^2 + (x_1^2)^2 = 60 + 90p \qquad (3-8)$$

which defines her total demand x_1^2 as a function of the realtive price p of the second good. Since her excess demand for good 1 is $x_1^2 - 60$, the aggregate excess demand of both people for good 1 will be zero if and only if

$$10p + x_1^2 - 60 = 0 \qquad (3-9)$$

where x_1^2 satisfies $(3-8)$.

Since $(3-9)$ implies that

$$x_1^2 = 60 - 10p \qquad (3-10)$$

substitution for x_1^2 from $(3-10$ into $(3-8)$ implies that

$$(60 - 10p)^2 = 100p \qquad (3-11)$$

or

$$p^2 - 13p + 36 = 0 \qquad (3-12)$$

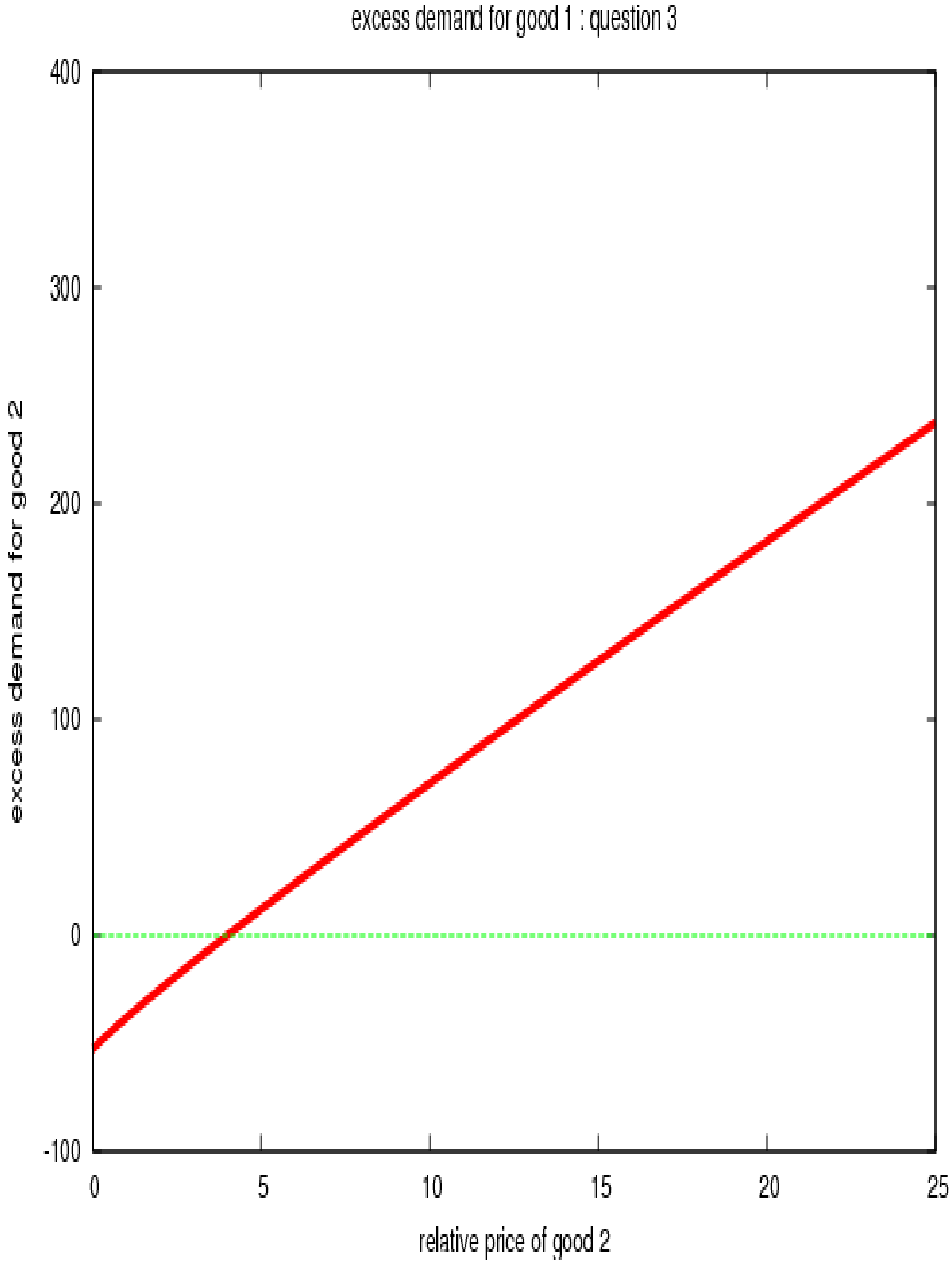
Equation $(3-12)$ has 2 roots : $p = 4$ and $p = 9$. [The same results could be obtained by solving $(3-8)$ for x_1^2 using the quadratic formula, and substituting this value of x_1^2 into the market clearing condition $(3-9)$.] But if $p = 9$, then equation $(3-10)$ implies that $x_1^2 = -30 < 0$, which makes no sense. The only relative prices which clear the market for good 1 are those price vectors (p_1, p_2) for which $p_2 = 4p_1$.

Walras's Law says that if the price vector (p_1, p_2) makes aggregate excess demand for good 1 equal to zero, then it must make aggregate excess demand for good 2 equal to zero. Therefore $(1, 4)$ (or some multiple of that price vector) is a price vector which clears the market.

The resulting equilibrium allocation is $\mathbf{x}^1 = (40, 20)$, $\mathbf{x}^2 = (20, 100)$. You can check that this allocation is Pareto optimal, using the equation of the contract curve in question #1.

The curve below graphs the excess demand for good 1 (as defined by equation $(3-9)$), when

x_1^2 satisfies $(3-8)$.



$Q4$. What are the core allocations to the economy described in question #2 above if, instead of 3 people, there were $3N$ people, N of each type, where N is large? (That is, all people have the same utility function, N of them have the endowment vector e^1, N of them have the endowment vector e^2, and N of them have the endowment vector e^3.)

$A4$. First, note that any allocation in the core must give the same utility to each person of a given type. That is, each type–1 person (all N of them) must get the same utility as each other.

Why? Let u^i be the **average** of the utility of each of the N people of type i. Let m^i be the minimum utility any person of type i gets. Now take the 3–person coalition, consisting of the worst–off person of each type. That coalition is just a $1/N$ microcosm of the whole economy. It can guarantee its three members the utility levels (u^1, u^2, u^3) simply by dividing up its endowments in the same way that the endowments of the whole coalition are divided up — on average — in the proposed allocation. So if $m^i < u^i$ for any type i, this new coalition of the worst–off people of each type can blcok the proposed allocation.

So any allocation in the core consists of three utility levels, one for each type. Any allocation must be Pareto efficient, so that (as in the answer to question 2), each person must get the same consumption of each good. So any allocation in the core must be defined by three combinations $(a, a), (b, b), (c, c)$, with $a + b + c = 3$, where each type–1 person gets (a, a), each type–2 person gets (b, b), and each type–3 person gets (c, c).

As in question 2, if the allocation is in the core, $c = 1$: each type–3 person can get utility of 1 on her own, and if she gets more than 1, then all the type–1 and type–2 people together can block the allocation.

How low can a be? As in question 2, suppose that $a < 1$, so that $b = 2 - a > 1$, and consider what a type–1 person could do.

Why just 1 person of type 1? Why not all N of them trying to get together a coalition to block the allocation which gives them each only (a, a) (with $a < 1$)? Well, if all N of them formed a coalition with all N type–3 people, this coalition would be exactly like the coalition of 1 type–1 person and 1 type–3 person used in question 2. That coalition is useful : it can block any allocation in which $a < 0.73$ (approximately). But with more than 1 person of each type, maybe more allocations can be blocked. And the best way to try and block an allocation in which $a < 1$ is to get a large number of people of type 3 (who have to offered a utility level of only 1). It's not a good strategy to try and get people of type 2 into this new blocking coalition, since each person of type 2 has to be offered a utility level of $b^2 > 1$; to join, a person of type 2 has to do better than he would in the original allocation which the coalition is trying to block.

But any individual type–1 person could form an $N + 1$–person coalition with the N people of type 3. This coalition has an aggregate endowment of $(N+2, N)$. So the (lone) type–1 person could propose giving each type–3 person $(\frac{N+2}{N}z, z)$, leaving her with $\frac{N+2}{N}q, q)$, with $Nz + q = N$. To get the type–3 people to join, the type–1 person must offer each of them a utility level of at least z. Therefore, the lowest z she can offer them is a z such that

$$\frac{N+2}{N}z^2 = 1$$

or

$$z = \sqrt{\frac{N}{N+2}} \qquad (4-1)$$

If z satisfies the constraint $(4-1)$ then

$$q = N(1-z) = N[1 - \sqrt{\frac{N}{N+2}} = \frac{N}{\sqrt{N+2}}[\sqrt{N+2} - \sqrt{N}] \qquad (4-2)$$

This new $N+1$–person coalition offers the lone type–1 person a utlity of $\frac{N+2}{N}q^2$, or

$$N(\sqrt{N+2} - \sqrt{N})^2 \qquad (4-3)$$

She will prefer this consumption bundle $\frac{N+2}{N}q, q)$ to the proposed bundle (a, a) only if it gives her utility greater than a^2. Therefore, this $N+1$ person coalition will block the proposed allocation only if

$$N(\sqrt{N+2} - \sqrt{N})^2 > a^2 \qquad (4-4)$$

So if the proposed allocation is in the core, equation $(4-4)$ cannot hold : the proposed allocation cannot be blocked by the $N+1$ person coalition if it is in the core. $(4-4)$ not holding is the same thing as the following inequality being true :

$$a \geq \sqrt{N^2 + 2N} - N \qquad (4-5)$$

Now since

$$\sqrt{N^2 + 2N} < \sqrt{N^2 + 2N + 1} = N + 1$$

the right hand side of inequality $(4-5)$ must be less than 1. But the right hand side of $(4-5)$ is an increasing function of N [why? its derivative with respect to N is $\frac{N+1}{\sqrt{N^2+2N}} - 1 > 0$].

Just as each person of type 1 must be offered a combination (a, a) satisfying $(4-5)$, similarly each person of type 2 must be offered a consumption bundle (b, b) with $b \geq \sqrt{N^2 + 2N} - N$, to prevent one of the type–2 people from blocking the allocation by forming an $N+1$–person coalition with the N type–3 people.

Therefore, an allocation will be in the core of this $3N$–person economy only if each type–1 person gets (a, a), each type–2 person gets (b, b), and each type–3 person gets $(1, 1)$, where $a+b = 2$, and where the minimum of a and b is at least

$$c_{min} \equiv \sqrt{N^2 + 2N} - N$$

As N grows large, c_{min} increases, and it approaches 1.

This property of c_{min} is probably best seen simply by calculating its value for a few different values of N. But to prove formally that $c_{min} \rightarrow 1$ as $N \rightarrow \infty$, let

$$A = N + 1$$

and

$$B = \sqrt{N^2 + N}$$

so that

$$1 - c_{min} = A - B \qquad (4-6)$$

Note that

$$A^2 - B^2 = (N+1)^2 - (N^2 + 2N) = 1 \qquad (4-7)$$

Also,

$$A^2 - B^2 = (A+B)(A-B) \qquad (4-8)$$

so that

$$A - B = \frac{A^2 - B^2}{A + B} = \frac{1}{N + 1 + \sqrt{N^2 + N}} \qquad (4-9)$$

As N grows large, the denominator on the right side of equation $(4-9)$ grows very large, so that $A - B$ must approach 0 as N grows large. But equation $(4-6)$ shows that $A - B$ approaching 0 is the same thing as c_{min} approaching 1.

The result here is a special case of a general result : if we "replicate" an economy (by "cloning" each person N times), then the core of the economy shrinks. As the economy grows large, the only allocations in the core are competitive equilibrium allocations. In this example, there is a unique competitive equilibrium allocation, in which each person's consumption bundle is $(1, 1)$.

*Q*5. Find all the pure–strategy Nash equilibria in the following strategic–form two–person game.

	LL	L	R	RR
tt	$(8,8)$	$(3,5)$	$(2,4)$	$(0,1)$
t	$(8,3)$	$(9,4)$	$(4,0)$	$(2,2)$
b	$(6,5)$	$(1,2)$	$(6,4)$	$(5,0)$
bb	$(4,4)$	$(0,8)$	$(3,8)$	$(10,7)$

*A*5. This game can be solved most easily by repeated elimination of strictly dominated strategies. First note that the strategy RR is strictly dominated by L for person 2 : $1 < 5$ and $2 < 4$ and $0 < 2$ and $7 < 8$. Whatever strategy person 1 plays, person 2 is better off playing column L

than column RR. So we can cross out column RR from the matrix : player 2 would never choose to use this strategy if he is rational.

With column RR eliminated, strategy bb is now strictly dominated by strategy t for player 1 : $4 < 8$ and $0 < 9$ and $3 < 4$. Knowing that player 2 is rational, and therefore never going to play strategy RR, player 1 realizes that startegy t is better for her than strategy bb, whatever of the 3 "sensible" strategies player 2 might happen to play. We can cross off row bb from the game matrix.

With bb gone, strategy LL now strictly dominates strategy R for player 2. We can cross out that strategy, leaving a game with 3 rows and 2 columns.

However, with R and RR gone, row b is now dominated (by tt or t) for player 1. So we can cross out row b, leaving a 2–by–2 matrix

	LL	L
tt	$(8, 8)$	$(3, 5)$
t	$(8, 3)$	$(9, 4)$

Player 1 now has a **weakly** dominated strategy : tt never gives a better payoff to her than t. Crossing that out leaves her with only t, and player 2's best response to t is L. So the game can be solved by repeated elimination of weakly dominated strategies : the solution is the pair of strategies (t, L).

However, repeated elimination of **strictly** dominated strategies gets us only as far as the 2–by–2 matrix above : there are no strictly dominated strategies in that game. And in fact, there is another Nash equilibrium to the game : if player 1 were to play tt, then 2's best response (to tt) is LL ; if player 2 plays LL, then 1 is indifferent between t and tt, so that tt is one of her best responses to LL. Therefore (tt, LL) is a Nash equilibrium to the original game — even though it involves a player (player 1) playing a strategy that is weakly dominated in the 2–by–2 game above to which the original game was reduced.

Even without elimination of any dominated strategies, it is straightforward to show that (tt, LL) and (t, L) are the only pure–strategy Nash equilibria to this 4–by–4 game in strategic form. But these are actually the only Nash equilibria in pure or mixed strategies : if a strategy can be crossed out during the process of repeated elimination of weakly dominated strategies, then it will never be played with positive probability in any mixed–strategy Nash equilibrium. Since all the strategies except for t and L get crossed off in repeated elimination of weakly dominated strategies, there are no Nash equilibria in which either player does any mixing.