$Q1$. Are the preferences described below strictly monotonic? Convex? Explain briefly.

There are two goods in the person's consumption bundle. In comparing any 2 bundles, $x = (x_1, x_2)$ and $y = (y_1, y_2)$, she gives points for a bundle which has more of a good. If $x_1 > y_1$, then bundle x gets 1 point; if $y_1 > x_1$ then bundle y gets 1 point ; if $x_1 = y_1$, then each bundle gets half a point. If $x_2 > y_2$, then bundle x gets 2 more points ; if $y_2 > x_2$, then bundle y gets 2 more points ; if $x_2 = y_2$, then each bundle gets 1 point.

(So, for example, if $x = (3, 2)$ and $y = (4, 1)$ then x would get 2 points and y would get 1 point.)

She finds bundle x at least as good as bundle y if and only if x gets at least as many points as y.

$A1$. Note first that the person is indifferent between bundles x and y if and only if each bundle gets the same number of points. As well, she prefers strictly one bundle to another if and only if the first bundle gets strictly more points.

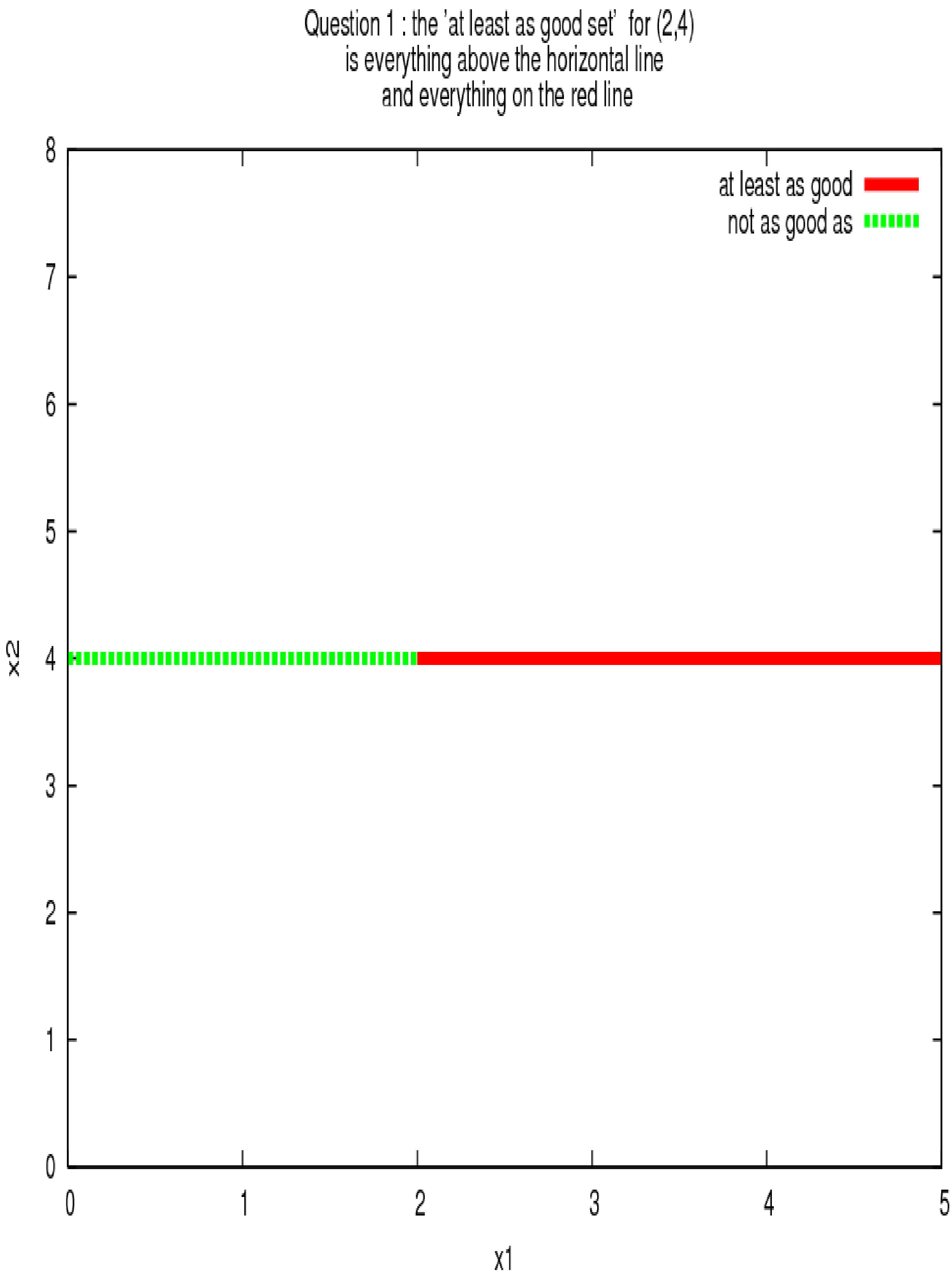
The preferences are strictly monotonic. If bundle x has strictly more of both goods (1 and 2) than bundle y, then bundle x gets 3 points, bundle y 0 points, and the person ranks bundle x strictly higher. If bundle x had exactly the same amount of one good, and strictly more of the other, then, again, bundle x would get more points (either 2 points to y's 1 if x has strictly more of good 1, or 2.5 points to y's 0.5 if x has strictly more of good 2). So a good which has more of one good, and at least as much of the other, must be preferred strictly, so that these preferences are monotonic.

Next, note that the only way a person can be indifferent between bundles x and y is if the two bundles are exactly the same. If bundle x had more of good 1, for example, and bundle y more of good 2, then bundle y would outscore bundle x by a 2–to–1 margin.

What bundles are considered at least as good as bundle x for this person? Suppose that $x = (2, 4)$ for concreteness. Any bundle y with $y_2 > 4$ will get a higher score than x. Any bundle y with $y_2 < 4$ will get lower score than x. So, in a diagram (with good 1 measured on the horizontal axis, good 2 on the vertical), any bundle which is above x will be preferred to x (whether it is to the right or to the left of x). Any bundle below x cannot be considered at least as good as x, whether it is to the right of x or to the left.

If $y_2 = x_2 = 4$, then bundle y gets a higher score than x if (and only if) $y_1 > x_1 = 2$. So any bundle which is to the right of x (but neither above nor below) is at least as good (strictly better, in fact). Any bundle which is to the left of x (but not above and not below) is not as good.

The "at least as good set" to any bundle x, then is the set of all bundles above x, and all bundles exactly to the right of x. That is a convex set : connect any 2 points in this set with a line, and you stay in the set.


Question 1 : the 'at least as good set' for (2,4)
is everything above the horizontal line
and everything on the red line
at least as good
not as good as
x2
x1
0
1
2
3
4
5
6
7
8

[However, the "at least as good set" is not closed : the bundles $(1,5),(1,4.5),(1,4.25),(1,4.125),\cdots$ are all at least as good as $x=(2,4)$, but the limit of that sequence $(1,4)$ is not at least as good. So these preferences are not continuous.]

$Q2$. Are the preferences represented by the utility function below strictly monotonic? Convex? Explain briefly.

$$u(x_1,x_2,x_3))=10-\frac{1}{x_1x_2x_3+1}$$

$A2$. Straightforward differentiation shows that the marginal utility of good 1 is

$$\frac{\partial u}{\partial x_1}=\frac{x_2x_3}{(x_1x_2x_3+1)^2}$$

This expression is non–negative, and is strictly positive whenever x_2 and x_3 are both strictly positive.

Similarly

$$\frac{\partial u}{\partial x_2}=\frac{x_1x_3}{(x_1x_2x_3+1)^2}$$

$$\frac{\partial u}{\partial x_3}=\frac{x_1x_2}{(x_1x_2x_3+1)^2}$$

so that preferences are strictly monotonic whenever (x_1,x_2,x_3) is strictly positive.

The final technicality to check is the behaviour of preferences when bundles contain a quantity 0 of some goods. Technically, it is not true here that bundle x is strictly preferred to bundle y if x has at least as much of all goods, and strictly more of 1 good. The consumer is indifferent between bundles $(2,1,0)$ and $(3,2,0)$, even though the first bundle has more of goods 1 and 2, and just as much of good 3.

However, if a bundle $x>>y$, then the consumer must prefer x strictly. If $y>>0$, that result follows from the fact that all partial derivatives of u are positive when consumption bundles have strictly positive quantities of each good ; if $y_i=0$ for some good, then $u(y)=9$, and $u(x)>9$ since $x>>0$. So preferences are strictly monotonic.

To check convexity, it is easiest to take a monotonic transformation of the utility function. Note that the value of $u(x_1,x_2,x_3)$ depends only on the value of $x_1x_2x_3$. So the utility function

$$U(x_1,x_2,x_3)=x_1x_2x_3$$

represents the same preferences as the function $u(x_1,x_2,x_3)=10-\frac{1}{x_1x_2x_3+1}$. [Formally,

$$u(x_1,x_2,x_3)=\Psi(U(x_1,x_2,x_3))$$

where

$$\Psi(z)=10-\frac{1}{z+1}$$

is a monotonically increasing transformation.]

And the function $U(x_1, x_2, x_3)$ is a Cobb–Douglas utility function, which means that it represents strictly convex preferences. For example

$$\tilde{U}(x_1, x_2, x_3) = \ln[U(x_1, x_2, x)] = \ln x_1 + \ln x_2 + \ln x_3$$

is a monotonically increasing transformation of U [and therefore a monotonically increasing transformation of u], and is a concave function.

*Q*3. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2) = \min\left(\ln x_1 + 2\ln x_2, 2\ln x_1 + \ln x_2\right)$$

(where "min" means "the minimum of").

As (I hope) the "minimum operator" in the definition of $u(x_1, x_2)$ suggests, there is a kink in this person's indifference curve, which is what makes this problem tricky.

What is the minimum of $\ln x_1 + 2\ln x_2$ and of $2\ln x_1 + \ln x_2$? It depends on which of those two expressions is bigger. Checking,

$$\ln x_1 + 2\ln x_2 > 2\ln x_1 + \ln x_2$$

if and only if

$$\ln x_2 > \ln x_2$$

if and only if

$$x_2 > x_1$$

So if $x_2 > x_1$, then $\ln x_1 + 2\ln x_2 > 2\ln x_1 + \ln x_2$, so that $u(x_1, x_2) = 2\ln x_1 + \ln x_2$, and if $x_1 > x_2$, then $\ln x_1 + 2\ln x_2 < 2\ln x_1 + \ln x_2$ so that $u(x_1, x_2) = \ln x_1 + 2\ln x_2$. [If $x_1 = x_2$, then $\ln x_1 + 2\ln x_2 = 2\ln x_1 + \ln x_2$, so that $u(x_1, x_2) = \ln x_1 + 2\ln x_2 = 2\ln x_1 + \ln x_2 = 3\ln x_1 = 3\ln x_2$.]

So what's the slope of an indifference curve?

If $x_2 > x_1$, then

$$u_1 = \frac{2}{x_1}$$

$$u_2 = \frac{1}{x_2}$$

so that

$$MRS = \frac{u_1}{u_2} = 2\frac{x_2}{x_1} \tag{3-1}$$

if $x_1 > x_2$,then

$$u_1 = \frac{1}{x_1}$$

$$u_2 = \frac{2}{x_2}$$

so that

$$MRS = \frac{u_1}{u_2} = \frac{x_2}{2x_1} \qquad (3-2)$$

At $x_1 = x_2$, the MRS falls discontinuously, as we move from left to right. Above (and to the left of) the line $x_1 = x_2$, expression $(3-1)$ says that the MRS approaches 2 as $x_1 \to x_2$. Below (and to the right), expression $(3-2)$ says that the MRS approaches 1/2.

So the indifference curves for these preferences are as drawn in figure 3. Above the 45–degree line, they are nice and smooth ; below the 45–degree line they are nice and smooth ; but there are kinks along the 45–degree line, as the slope falls (in absolute value) from 2 to 1/2.

This person's indifference curves never have a slope between 1/2 and 2 (in absolute value). Above the 45–degree line, the slope is 2 or greater ; below the 45–degree line the indifference curves have a slope of 1/2 or less.

So what happens if the price ratio is between 1/2 and 2? The person's indifference curves can't be tangent to a budget line above the 45–degree line ; the indifference curves are too steep. And they can't be tangent below the 45–degree line ; the indifference curves are not steep enough. What happens is an indifference curve is "tangent" to the budget line at a kink along the 45–degree line (as in the figure).

Along the 45–degree line, $x_1 = x_2$. So — if p_1/p_2 is between 1/2 and 2 — the person chooses a bundle along the 45–degree line, where $x_1 = x_2$. Her budget constraint then requires that

$$p_1 x_1 + p_2 x_2 = (p_1 + p_2) x_1 = m$$

so that her demands for the two goods are

$$x_1^m(p_1, p_2, m) = x_2^m(p_1, p_2, m) = \frac{m}{p_1 + p_2} \quad \text{if} \quad \frac{1}{2} \leq \frac{p_1}{p_2} \leq 2 \qquad (3-3)$$

If $p_1 > 2p_2$, then she will pick a point on her budget line where her MRS is greater than 2 ; that must be above the 45–degree line, because only there can she have an MRS greater than 2. Above the 45–degree line,

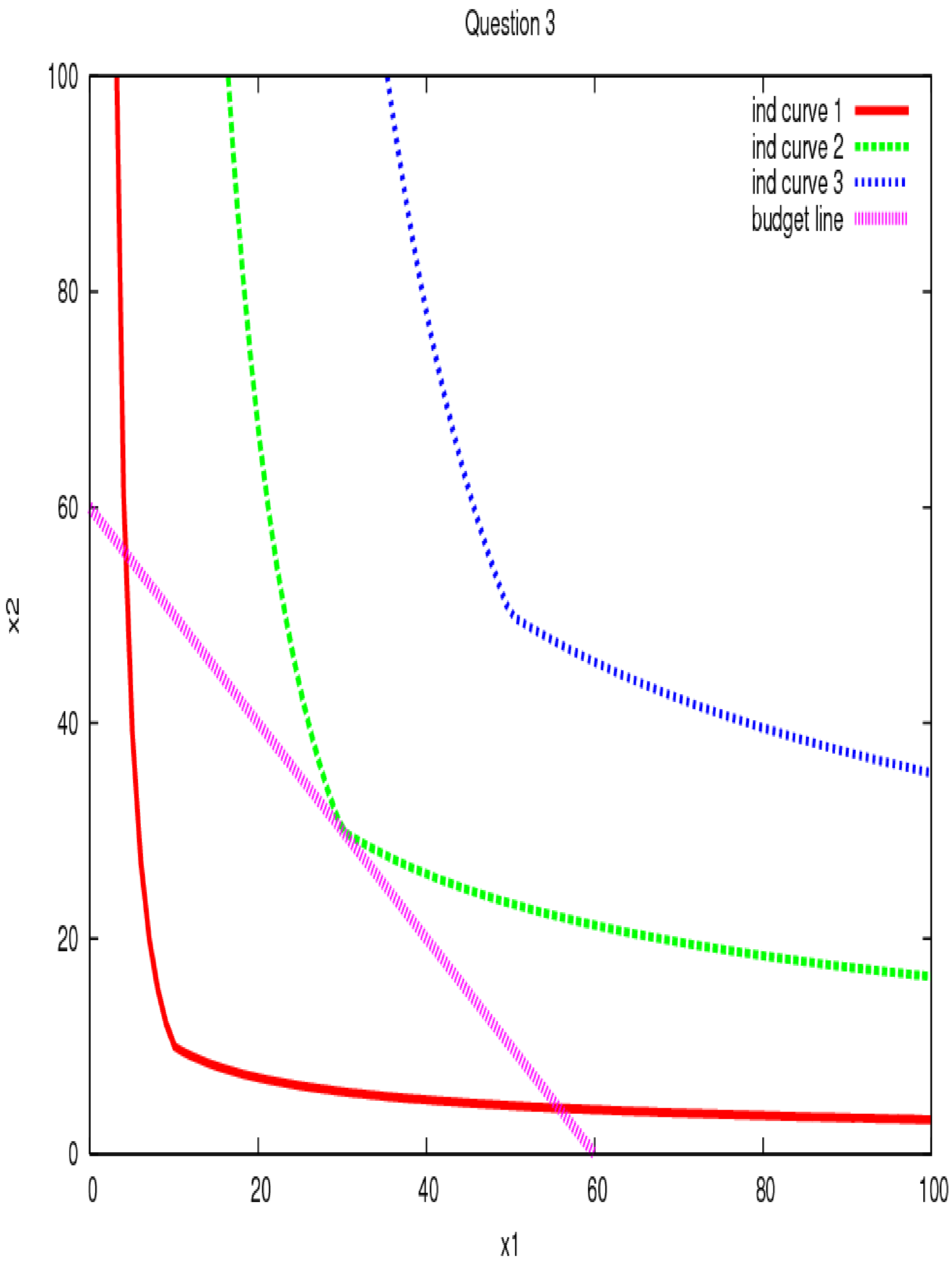
$$u(x_1, x_2) = 2 \ln x_1 + \ln x_2$$

which define Cobb–Douglas preferences. So her demands for the two goods would be

$$x_1^m(p_1, p_2, m) = \frac{m}{3p_1}; x_2^m(p_1, p_2, m) = \frac{2m}{3p_2} \quad \text{if} \quad p_1 > 2p_2 \qquad (3-4)$$

Similarly, if $p_2 > 2p_1$, she will pick a point on her budget line which is below the 45–degree line, where her MRS is 1/2 or less. Here $u(x_1, x_2) = 2 \ln x_1 + \ln x_2$, again Cobb–Douglas preferences, so that

$$x_1^m(p_1, p_2, m) = \frac{2m}{3p_1}; x_2^m(p_1, p_2, m) = \frac{m}{3p_2} \quad \text{if} \quad p_2 > 2p_1 \qquad (3-5)$$



Question 3
100
80
60
40
20
0
0
20
40
60
80
100
x2
x1
ind curve 1
ind curve 2
ind curve 3
budget line

Equations $(3-3)$, $(3-4)$ and $(3-5)$ define completely her Marshallian demands for these "kinked" preferences.

*Q*4. Calculate a person's Marshallian demand functions, her indirect utility function, her Hicksian demand functions, and her expenditure function, if her direct utility function is

$$u(x_1, x_2, x_3) = x_1 + \ln x_2 + 2\sqrt{x_3}$$

*A*4. Here the person's preferences are quasi–linear, so that her Marshallian demands for goods 2 and 3 should not depend on her income.

From the first–order conditions for utility maximization

$$\frac{\partial u}{\partial x_1} = 1 = \lambda p_1 \qquad (4-1)$$

$$\frac{\partial u}{\partial x_2} = \frac{1}{x_2} = \lambda p_2 \qquad (4-2)$$

$$\frac{\partial u}{\partial x_3} = \frac{1}{\sqrt{x_3}} = \lambda p_3 \qquad (4-3)$$

Substitution for λ from equation $(4-1)$ into equations $(4-2)$ and $(4-3)$ yields

$$x_2 = \frac{p_1}{p_2} \qquad (4-4)$$

$$x_3 = [\frac{p_1}{p_3}]^2 \qquad (4-5)$$

which are the Marshallian demand functions for goods 2 and 3.

Substitution of $(4-4)$ and $(4-5)$ into the budget constraint $p_1x_1 + p_2x_2 + p_3x_3 = y$ implies

$$p_1x_1 + p_1 + \frac{(p_1)^2}{p_3} = y \qquad (4-6)$$

or

$$x_1 = \frac{y}{p_1} - 1 - \frac{p_1}{p_3} \qquad (4-7)$$

which is the Marshallian demand for good 1. [This expression is only valid if income is large enough so that $y > p_1 + [\frac{p_1}{p_3}]^2$.]

Substitution from $(4-4)$, $(4-5)$ and $(4-7)$ back into the direct utility function implies that

$$u = \frac{y}{p_1} - 1 - \frac{p_1}{p_3} + \ln\frac{p_1}{p_2} + 2\frac{p_1}{p_3}$$

or

$$v(\mathbf{p}, y) = \frac{y}{p_1} - 1 + \ln p_1 - \ln p_2 + \frac{p_1}{p_3} \qquad (4-8)$$

which is the person's indirect utility function.

Using the fact that $v(\mathbf{p}, e(\mathbf{p}, u)) = u$, $(4-8)$ implies that

$$e(\mathbf{p}, u) = p_1 u + p_1 - p_1 \ln p_1 + p_1 \ln p_2 - \frac{(p_1)^2}{p_3} \qquad (4-9)$$

Now the Hicksian demand functions can be obtained as the partial derivatives of the expenditure function with respect to the prices :

$$x_1^h(\mathbf{p}, u) = u - \ln p_1 + \ln p_2 - 2\frac{p_1}{p_3} \qquad (4-10)$$

$$x_2^h(\mathbf{p}, u) = \frac{p_1}{p_2} \qquad (4-11)$$

$$x_3^h(\mathbf{p}, u) = [\frac{p_1}{p_3}]^2 \qquad (4-12)$$

The quasi-linearity of the direct utility function implies that the Hicksian demands for goods 2 and 3 are identical to the Marshallian demands.

$Q5$. Derive the Slutsky matrix (that is , the 2–by–2 matrix of derivatives of Hicksian demands with respect to prices) for a consumer whose preferences can be represented by the direct utility function

$$u(x_1, x_2) = x_1 + \ln x_2$$

$A5$. Solving first the expenditure minimization problem which defines the Hicksian demands, the first–order conditions are

$$p_1 = \mu \frac{\partial u}{\partial x_1} = \mu \qquad (5-1)$$

$$p_2 = \mu \frac{\partial u}{\partial x_2} = \frac{\mu}{x_2} \qquad (5-2)$$

Substituting for μ from $(5-1)$ into $(5-2)$ gives

$$x_2^h(p_1, p_2, u) = \frac{p_1}{p_2} \qquad (5-3)$$

the Hicksian demand function for good 2. Substituting back from $(5-3)$ into the definition of the utility function

$$u = x_1 + \ln p_1 - \ln p_2 \qquad (5-4)$$

or

$$x_1^h(p_1, p_2, u) = u - \ln p_1 + \ln p_2 \qquad (5-5)$$

Taking partial derivatives of $(5-3)$ and $(5-5)$ with respect to the prices

$$E_{22} = -\frac{p_1}{(p_2)^2}$$

$$E_{21} = \frac{1}{p_2}$$

$$E_{12} = \frac{1}{p_2}$$

$$E_{11} = -\frac{1}{p_1}$$

So that the Slutsky matrix is

$$\begin{pmatrix} -\frac{1}{p_1} & \frac{1}{p_2} \\ \frac{1}{p_2} & -\frac{p_1}{(p_2)^2} \end{pmatrix}$$

which is a symmetric, negative semi–definite matrix.