*Q*1. Explain why the following three equations cannot be the Marshallian demand functions of a consumer with well–behaved preferences, even when $p_1 \geq p_3$.

$$x_1(\mathbf{p}, y) = \frac{y}{2p_1}$$

$$x_2(\mathbf{p}, y) = \frac{p_3 y}{2p_1 p_2}$$

$$x_3(\mathbf{p}, y) = \frac{(p_1 - p_3)y}{2p_1 p_3}$$

*A*1. These functions do exhibit some of the properties of Marshallian demand functions. They are each homogeneous of degree 0 in prices and income together : multiplying all 3 prices by some constant t, and multiplying income by the same constant t, leaves each of the 3 demands unchanged.

They also obey "budget balancedness" : $p_1 x_1(\mathbf{p}, y) + p_2 x_2(\mathbf{p}, y) + p_3 x_3(\mathbf{p}, y) = y$, so that the "demands" are on the boundary of the consumer's budget set (whatever are the prices or income).

But the Slutsky matrix must also be symmetric, for demand functions arising from utility maximization by consumers with well–behaved preferences. It must be the case that $S_{ij} = S_{ji}$ for each pair of goods i and j, where the substitution terms S_{ij} can be computed from the Marshallian demand functions by

$$S_{ij} = \frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i^M}{\partial y}$$

In this case, for example,

$$S_{12} = \frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1^M}{\partial y} = \frac{p_3 y}{(p_1)^2 p_2}$$

and

$$S_{21} = \frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2^M}{\partial y} = -\frac{p_3 y}{(p_1)^2 p_2} + \frac{p_3 y}{(p_1)^2 p_2} = 0 \neq S_{12}$$

so that Theorem 1.16 of the textbook does not hold.

*Q*2. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at three different times, and the quantities x^t of the 3 goods chosen at the three different times. (For example, the second row indicates that the consumer chose the bundle $\mathbf{x} = (24, 21, 20)$ when the price vector was $\mathbf{p} = (2, 1, 2)$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	1	2	2	22	20	23
2	2	1	2	24	21	20
3	2	2	1	21	23	21

*A*1. It is probably best to compute the cost of each bundle, in each year. So, in the matrix below, the i–th element in the j–th row indicates the cost of bundle i at year–j prices.

$$\begin{matrix} 108 & 106 & 109 \\ 110 & 109 & 107 \\ 107 & 110 & 109 \end{matrix}$$

(So, for example, the bundle $\mathbf{x}^2$, which was chosen in year 2, would cost 106 using year–1 prices $\mathbf{p}^1$.)

The bundle chosen in year t is revealed directly to be preferred to some other bundle $\mathbf{x}^s$ if (and only if) the person could afford the other bundle $\mathbf{x}^s$ at year–t prices. That will occur if and only if the expenditure in column s of row t is no bigger than the expenditure on the diagonal, in row t.

Therefore, $\mathbf{x}^1$ is directly revealed preferred to $\mathbf{x}^2$ by the first row of the matrix : the person chose $\mathbf{x}^1$, costing 108, in year 1, when she could have afforded bundle $\mathbf{x}^2$, which would have cost only 106. $\mathbf{x}^1$ is **not** directly revealed preferred to $\mathbf{x}^3$, since the person could not have afforded $\mathbf{x}^3$ in year 1.

Row 2 shows that $\mathbf{x}^2$ is directly revealed preferred to $\mathbf{x}^3$ — but not to $\mathbf{x}^1$. Row 3 shows that $\mathbf{x}^3$ is directly revealed preferred to $\mathbf{x}^1$, but not to $\mathbf{x}^2$.

Therefore, there are no violations of the weak axiom of revealed preference in these data : for any pair i and j, if $\mathbf{x}^i$ is directly revealed preferred to bfx^j, then bfx^j is not directly revealed preferred to $\mathbf{x}^i$.

However, the strong axiom of revealed preference is violated in these data, since $\mathbf{x}^1$ is directly revealed preferred to $\mathbf{x}^2$ which is directly revealed preferred to $\mathbf{x}^3$ which is directly revealed preferred to $\mathbf{x}^1$.

*Q*3. Frank and Ernest are both risk–averse expected utility maximizers. Frank has a utility–of–wealth function

$$U(W) = \ln W$$

while Ernest has a utility–of–wealth function

$$V(W) = 1 - e^{-W}$$

Give an example of a simple ("2 state") gamble which Ernest would accept but which Frank would reject, and an example of another simple ("2 state") gamble which Frank would accept but which Ernest would reject.

*A*3. There are many possible examples. But these examples can be constructed most easily by looking at the shapes of the two utility–of–wealth functions. Frank's utility of wealth approaches $-\infty$ as W approaches 0. So he will tend to reject any gamble which may lead to some chance of him losing nearly all of his wealth. Ernest's utility of wealth, on the other hand, is bounded above by 1 as $W \to \infty$. So he will tend to reject gambles which offer a small probability of a vast fortune.

For concreteness, suppose that Frank and Ernest both have an initial wealth of 2, so that Frank's initial utility is $\ln W \approx 0.693$ and Ernest's initial utility is $1 - e^{-2} \approx 0.865$.

The first gamble offers a large wealth W' with probability 0.5, and a reduced wealth of 1, with probability 0.5. Frank will accept this gamble if W' is large enough. His expected utility from the gamble is $(0.5)\ln 1 + (0.5)\ln W'$. Since $\ln 1 = 0$, Frank will accept this gamble, provided that $W' > 4$, since $\ln W'$ gets very large if W' is large enough. On the other hand, Ernest's expected utility from the gamble is

$$(0.5)(1 - e^{-1}) + (0.5)(1 - e^{-W'}) = (0.5)(0.632) + (0.5)(1 - e^{-W'})$$

No matter how large W' is, $1 - e^{-W'}$ must be less than 1. So the above expression must be less than $1.632/2$ which is less than Ernest's initial utility of 0.865.

Hence, any gamble which offers 1 with probability 0.5 and $W' > 4$ with probability 0.5 will be accepted by Frank but rejected by Ernest.

The second gamble offers a small probability of a very bad outcome, which scares Frank much more than Ernest (since Frank's utility–of–wealth function gets infinitely steep near a wealth of 0, and Ernest's does not).

For example, consider a gamble which offers a wealth of 5 with probability 0.95 and a wealth of 0.00000001 with probability 0.05. Frank's expected utility from the gamble is

$$(0.05)(\ln 0.00000001) + (0.95)(\ln 5) = 0.608 < 0.693$$

so that he will reject it. Ernest's expected utility is

$$(0.05)(1 - e^{-0.00000001}) + (0.95)(1 - e^{-5}) = 0.9436 > 0.865$$

so that he will accept the bet.

*Q*4. Suppose that an expected utility maximizer has a utility–of–wealth function

$$U(W) = \frac{1}{1-\beta} W^{1-\beta} \quad \beta < 1 \quad \beta \neq 0$$

The person has an initial wealth of W_0. She has the opportunity to risk all her initial wealth on the toss of a fair coin. If the coin lands "tails", she would lose all her wealth. If the coin lands "heads", she would collect a multiple A of her initial wealth, where $A > 2$. (This is an "all or nothing" proposition. She must bet W_0 if she bets.)

For what values of initial wealth W_0, and of the parameter β, would the person be willing to participate in this risky opportunity?

*A*4. Her expected utility, if she takes the risky opportunity, is

$$EU_1 = (0.5)\frac{1}{1-\beta}A^{1-\beta}W^{1-\beta}$$

and her expected utility if she stays with her initial wealth is

$$EU_0 = \frac{1}{1-\beta}W^{1-\beta}$$

Comparing the 2 expressions, she will take the opportunity only if

$$(0.5)A^{1-\beta} > 1 \qquad (4-1)$$

So whether or not she accepts the opportunity depends only on her risk aversion parameter β and not at all on her initial wealth W. (That is because she has a constant coefficient of relative risk aversion, and the opportunity offers either a proportional increase in her wealth, or a proportional decrease.)

Expression $(4-1)$ can be written

$$A^{1-\beta} > 2 \qquad (4-2)$$

or (taking the natural logarithm of both sides of $(4-2)$)

$$(1-\beta)\ln A > \ln 2 \qquad (4-3)$$

so that the values of her coefficient of relative risk aversion β for which she is willing to undertake the opportunity, are those for which

$$\beta < 1 - \frac{\ln 2}{\ln A} \qquad (4-4)$$

In particular, a person with a coefficient of relative risk aversion of 1 or more will never be willing to take this risk, no matter how great the potential increase A in her wealth.

*Q*5. Suppose that there were two states of the world, good and bad, and three assets. Asset 1 pays a net return of 10 percent in either state of the world. Asset 2 pays a net return of 20 percent in the bad state, and 0 in the good state. Asset 3 pays a net return of r percent in the good state, and 0 in the bad state.

The probability of the good state is π and of the bad state is $1-\pi$.

The person can allocate her wealth among the 3 assets, but must have a non–negative amount invested in each assets (that is, she is not allowed to sell an asset short).

If the person is a risk–averse expected utility maximizer, for what values of r will she choose to invest positive amounts in all 3 assets?

*A*5. Notice that the question did not mention the probability of the good or bad state occurring. That is because the answer does not require this information.

First of all, note that if $r > 0.2$, then the person will never want to allocate any of her wealth into the safe asset. Why? A combination of 50 cents in asset 2, and 50 cents in asset 3 dominates an investment of a dollar in asset 1. This combination yields her a return of 10 cents per dollar in the good state, and $r/2$ in the bad state.

So if $r > 0.2$, and if she invested a positive amount in the safe asset, she is not maximizing her expected utility. Reducing her investment in asset 1 by $2x$ (where x is any small positive amount) and increasing her investment in asset 2 and in asset 3 by x each, will leave her return in the good state unchanged, and will raise her earnings in the bad state by $(r - 0.20)/2x$, so that the original investment plan could not have been optimal.

What if $r < 0.2$? Would it ever by optimal for her to invest positive amounts of money in all 3 assets? Suppose that she did put positive amounts of money in all 3 assets. Now consider her changing her plan slightly, this time *increasing* her investment in asset 1 by $2x$, and *reducing* her investment in assets 2 and 3 by x each. Now this adjustment of her portfolio leaves unchanged her return in the good state, and increases her earnings in the bad state by $(0.20 - r)/2x$. Hence the original plan could not have maximized her expected utility, since this modification increases her wealth in the bad state and leaves it unchanged in the good state.

Therefore, the only possible situation in which she would be willing to invest in all 3 assets would be if $r = 0.2$. In this case, actually, there will be many different allocations which maximize her expected utility : shifting $2x$ asset 1, and x each out of assets 2 and 3 will have no affect at all on her wealth in either state, if $r = 0.2$.

How could this answer be seen from solving the expected–utility maximization problem? Consider the problem of maximizing

$$\pi U[(1 + 0.2x_2 + 0.1(1 - x_2 - x_3))W] + (1 - \pi U[(1 + rx_2 + 0.1(1 - x_2 - x_3))W]$$

with respect to the proportions x_2 and x_3 of her wealth to put in assets 2 and 3, where π is the probability of the good state, and $U[\cdot]$ is her utility–of–wealth function.

The first–order conditions with respect to x_2 and x_3 are consistent with each other only if $r = 0.2$; if $r \neq 0.$, then the first–order condition with respect to x_2 and the first–order condition with respect to x_3 cannot both hold.

Does the probability π of the good state matter at all? Yes, it does matter for how much the person invests in each asset. If $r > 0.2$, the person puts all her money into assets 2 and 3 ; the higher is π, the more she invests in asset 2, and the less in asset 3. If $r < 0.2$, then the person invests only in the safe asset 1, and in one of the other two risky assets. If π is low, the only risky asset in which she invests is asset 3 ; if π is high, then the only risky asset in which she invests is asset 2.