GS/ECON 5010 Answers to Assignment 3 November 2007

$Q1$. What is the equation of the supply curve of a firm which has a total cost function with the equation

$$TC(q) = (q-6)^3 + 3q + 216$$

where q is the quantity of output produced by the firm?

$A1$. Given the total cost function, the marginal cost for the firm is

$$MC(q) = 3(q-6)^2 + 3 \qquad (1-1)$$

Equation $(1-1)$ defines a U–shaped curve, which reaches a minimum (of $MC = 3$) at $q = 6$.

The firm's supply curve is the upward–sloping part of its marginal cost curve, above its intersection with the average cost curve.

So the profit–maximizing relation $p = MC$, and equation $(1-1)$, imply a supply curve with the equation

$$p = 3(q-6)^2 + 3$$

or

$$q = \sqrt{\frac{p-3}{3}} + 6 \qquad (1-2)$$

Equation $(1-2)$ defines the firm's supply curve, but only for values of the price which are above the minimum average cost.

To find the minimum average cost, divide the total cost by q to get

$$AC(q) = \frac{(q-6)^3 + 216}{q} + 3 \qquad (1-3)$$

From equations $(1-1)$ and $(1-3)$, the marginal cost equals the average cost when

$$3(q-6)^2 q = (q-5)^3 + 216 \qquad (1-4)$$

or (expanding both sides of $(1-4)$)

$$3q^3 - 36q^2 + 108q = q^3 - 18q^2 + 108q - 216 + 216$$

which is equivalent to

$$2q(q-9) = 0 \qquad (1-5)$$

so that $MC(q) = AC(q)$ at $q = 9$.

Checking, from equation $(1-3)$,

$$AC'(q) = \frac{1}{q^2}[3q(q-6)^2 - (q-6)^3 - 216] = \frac{2(q+3)(q-6)^2 - 216}{q^2} \qquad (1-6)$$

From $(1-6)$, $AC'(q) < 0$ for small values of q, and $AC'(q) = 0$ at $q = 9$.

At $q = 9$, $MC = AC = 30$.

So the firm's supply curve is $q = 0$ if $p \leq 30$ amd $q = \sqrt{\frac{p-3}{3}} + 6$ if $p \geq 30$.

[Note that this is a long–run supply curve ; the firm's total cost equals 0 when $q = 0$ here so that there are no fixed costs.]

*Q*2. What is the equation of the long–run supply curve for an industry, if it has one firm of type t, for each value of $t = 1, 2, 3, \cdots, \infty$, and if a firm of type t has a long–run total cost function

$$TC(q) = (q-6)^3 + tq + 216$$

where q is the quantity of output produced by the firm?

*A*2. The total cost function in the first question is just a special case of the total cost function in this question, with $t = 3$.

So, from the answer to the first–question, a firm of type t has a U–shaped average cost curve, with a minimum at $q = 9$, and a minimum average cost (at $q = 9$) of $27 + t$.

Also, the supply curve for a type–t firm is $q = 0$ for $p \leq 27 + t$, and $q = \sqrt{\frac{p-t}{3}} + 6$ if $p \geq 27 + t$.

So a type–1 firm will produce a positive level of output only if the price is 28 or more, a type–2 firm will produce only if the price is 29 or more, and so on.

This means that if the price is less than 28, no firms will produce anything ; the price is below each firm's minimum average cost, and no output will be supplied. If the price is between 28 and 29, 1 firm (the type–1 firm) will produce, and it will produce $\sqrt{\frac{p-1}{3}} + 6$ units. If the price is between 29 and 30, 2 firms will produce, and their total output will be $\sqrt{\frac{p-1}{3}} + \sqrt{\frac{p-2}{3}} + 12$.

In general, if the price is between k and $k+1$, where k is an integer greater than 27, then $k - 27$ firms will produce, and the industry's total quantity supplied will be

$$6(k-27) + \sum_{i=1}^{k-27} \sqrt{\frac{p-i}{3}}$$

*Q*3. Suppose that a (single–price) monopoly faced a market demand with inverse demand curve $p(Q)$ (with $p'(Q) < 0$), and could produce under constant returns to scale at a marginal cost of c per unit produced.

However, the marginal cost c depends on the firm's fixed investment F in technology ; $c'(F) < 0$, since more investment in technology leads to lower–cost production techniques.

(i) Derive an expression for the firm's profit–maximizing level of technology investment.

(ii) Would consumers, in aggregate, be able to bribe the monopoly to invest in further technology investment, above the level which maximizes profits? Explain briefly.

$A3$. (i) Let $q^*(c)$ denote the profit–maximizing output level for a monopoly with a constant marginal cost of c, that is the solution to the equation

$$p(q^*[c]) + p'(q^*[c])q^*[c] = c \tag{3-1}$$

The monopoly's profit, ignoring its fixed cost of technology, would be

$$\pi(c) = p(q^*[c])q^*(c) - cq^*[c] \tag{3-2}$$

Differentiating equation $(3-2)$,

$$\pi'(c) = \left[p'(q^*[c]q^*[c] + p(q^*[c]) - c\right]\frac{\partial q^*}{\partial c} - q^*[c] \tag{3-3}$$

But equation $(3-1)$ implies that the term in large square brackets in $(3-3)$ is zero — that's the envelope theorem again — so that $\pi' = -q^*$.

Now consider the monopoly's choice of technology. It wants to choose an investment F in technology to maximize its overall profit

$$\pi(c(F)) - F \tag{3-4}$$

implying a first–order condition

$$-c'(F)q^*(c(F)) = 1 \tag{3-5}$$

That is, one dollar more invested in technology should raise its profits (net of fixed costs) π by one dollar.

(ii) The aggreate benefit to consumers can be measured as the area underneath the aggregate demand curve, and above the price. This "consumers' surplus" is

$$CS = \int_0^{q*} p(q)dq - p(q^*)q^*$$

(that is, the difference between total consumer surplus, and the total amount paid to the monopoly) if the firm produces q^* units of output.

So (since the derivative of an integral is the function itself)

$$\frac{\partial CS}{\partial q^*} = -p'(q^*)q^* > 0 \tag{3-6}$$

At the monopoly's profit–maximizing level of technology investment — the solution to equation $(3-5)$ — consumers could get the monopoly to invest more in cost–saving technology if they could get together and coordinate a campaign to bribe the firm. At the margin (that is, at the level of F for which $-c'(F)q^*(F) = 1$) a slight increase in technology investment has no effect on the monopoly's profit, since the cost savings are just offset by the expense of the added technology ivestment. But since the lower marginal cost would result in greater output and a lower price, consumers would benefit.

So the monopoly invests "too little" in technology, in that a further technology improvement would benefit consumers more than it would harm the firm.

*Q*4. Find every Cournot–Nash equilibrium in a duopoly, in which the demand function for the homogeneous product has the equation

$$Q = 14 - p$$

and in which both firms have the same total cost function,

$$\begin{aligned} TC(q) &= 12 + 2q \quad q > 0 \\ &= 0 \quad q = 0 \end{aligned}$$

*A*4. The fixed cost (of 12) does not matter for each firm's reaction function — provided that the firm does decide to produce a positive level of output.

So formula (4.15) of *Jehle* & *Reny* can just be plugged in here, implying an output level for each firm of

$$q_1 = q_2 = \frac{14 - 2}{3} = 4 \tag{4 − 1}$$

since here $a = 14$, $b = 1$, $c = 2$, $J = 2$.

But $q_1 = q_2 = 4$ will be an equilibrium only if each firm is making non–negative profits ; otherwise the firm would prefer to produce nothing at all (at a total cost of 0). When $q_1 = q_2 = 4$, then $p = 14 - (q_1 + q_2) = 6$, so that each firm's revenue is $6 \cdot 4 = 24$. The firm's total costs of production are $2(4)+15 = 23$, so that each firm is making positive profits. The output combination $q_1 = q_2 = 4$ is a Cournot–Nash equilibrium.

But here, becasue of the positive fixed costs, $q_1 = q_2 = 4$ is not the only Cournot–Nash equilibrium. If q_1 is high enough, then firm 2 would be unable to cover its fixed costs, no matter what it does. For example, if $q_1 = 8$, equation (4.14) of *Jehle* & *Reny* shows that firm 2's best output would be $q_2 = 2$, If $q_1 = 8$ and $q_2 = 2$, then $p = 4$, and firm 2's net profit would be $2(4) - 2(2) - 15 = -11 < 0$.

So a high output level, such as $q_1 = 8$ would deter entry by the other firm : if $q_1 = 8$, then firm 2's best reaction would be to produce nothing.

However $(8, 0)$ is not a Cournot–Nash equilibrium here, becasue firm 1 would not choose to produce 8 units of output. If firm 2 produces nothing, firm 1's output must be its own *best reaction* to $q_2 = 0$. Here that best reaction is $q_1 = 6$. You can see that $q_1 = 6$ and $q_2 = 0$ satisfies (4.14) for $j = 1$. In fact, $q_1 = 6$ is exactly the output firm 1 would choose if it were a single–price monopoly – which it is, if firm 2 chooses not to produce.

So is $(6, 0)$ a Cournot–Nash equilibrium here? Certainly firm 1 will want to produce $q_1 = 6$ if $q_2 = 0$. What will firm 2 want to do if $q_1 = 6$? Equation (4.14) shows that firm 2's best choice of output (other than 0) when $q_1 = 6$ is $q_2 = 3$. If $q_1 = 6$ and $q_2 = 3$, then $p = 5$, and firm 2 would earn profits of $5(3) - 2(3) - 15 = -6$. Firm 2 makes a loss, and would rather produce 0.

So $q_1 = 6, q_2 = 0$ is also a Cournot–Nash equilibrium. Firm 1 wants to produce 6 when firm 2 produces 0, and firm 2 wants to produce 0 when firm 1 produces 6.

Why should firm 1 be the lucky one? There is also another Cournot–Nash equilibrium, in which $q_1 = 0$ and $q_2 = 6$.

Therefore, there are 3 Cournot–Nash equilibria with this cost function and demand function : $(4,4), (6,0), (0,6)$.

*Q*5. What would be the equilibrium in a duopoly, if the demand and cost functions were those of question #4 above, but in which the firms moved **sequentially** (as in the "Stackelberg duopoly" presented in question 4.9 of the text)? That is, firm 1 moves first, committing to a quantity of output which it will produce. Firm 2 then observes firm 1's quantity, and, after doing so, chooses its own quantity. (Here firm #1 is aware that firm #2 will be choosing its output level last, and will be reacting to firm #1's own choice of quantity.)

*A*5. When firm 1 moves first, in can anticipate how its rival will react to its choice of output. In this case, firm 2 will choose to produce

$$q_2^r = 6 - \frac{q_1}{2} \tag{5 − 1}$$

if it chooses to produce anything at all. Equation $(5-1)$ is firm 2's reaction function, and can be derived from (4.14) of *Jehle* & *Reny* with $j = 2$.

But firm 2 will only choose the output level defined by $(5-1)$ if it will make non–negative profit ; otherwise it will choose to produce nothing.

So if it were not for the fixed cost, firm 1 would want to choose its output q_1 so as to maximize its own profits

$$[a - b(q_1 + q_2^r(q_1)]q_1 - C(q_1) \tag{5 − 2}$$

an expression which takes into account the fact that more output by firm 1 will lead to less output by firm 2.

[When $C(q_1) = cq_1$, then maximizing $(5-2)$ actually leads to an optimal choice for firm 1 of

$$q_1^* = \frac{a-c}{2b} \tag{5 − 3}$$

which would be the answer to the question if there were no fixed costs.]

But here, the problem is actually easy for firm 1 to solve. From the solution to question #4 above, it knows that if it chooses an output of $q_1 = 6$, that firm 2 would then react by choosing to produce nothing. That is, an initial choice of $q_1 = 6$ will deter entry by firm 2. But $q_1 = 6$ is also the profit–maximizing output for firm 1 to choose if it does not need to worry about another firm.

So the equilibrium here, when firm 1 has a *first–mover advantage* is for firm 1 to choose $q_1 = 6$, and for firm 2 to react be choosing not to produce at all.