$Q1$. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of 10 units of good 1 and 30 units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = \ln x_1^1 + \ln x_2^1$$

$$u^2(x_1^2, x_2^2) = 112 - \frac{1}{(x_1^2)^2} + \ln x_2^2$$

where x_j^i is person i's consumption of good j?

$A1$. Given the people's utility functions, their marginal rates of substitution are

$$MRS^1 \equiv \frac{u_1^1}{u_2^1} = \frac{x_2^1}{x_1^1} \qquad (1-1)$$

$$MRS^2 \equiv \frac{u_1^2}{u_2^2} = \frac{2x_2^2}{(x_1^2)^3} \qquad (1-2)$$

An allocation $(x_1^1, x_2^1, x_1^2, x_2^2)$ will be on the contract curve if $MRS^1 = MRS^2$. Since the total endowment of good 1 is 10, and the total endowment of good 2 is 30, therefore

$$x_1^2 = 10 - x_1^1$$

$$x_2^2 = 30 - x_2^1$$

so that an allocation is on the contract curve if

$$\frac{x_2^1}{x_1^1} = \frac{2(30 - x_2^1)}{(10 - x_1^1)^3} \qquad (1-3)$$

or

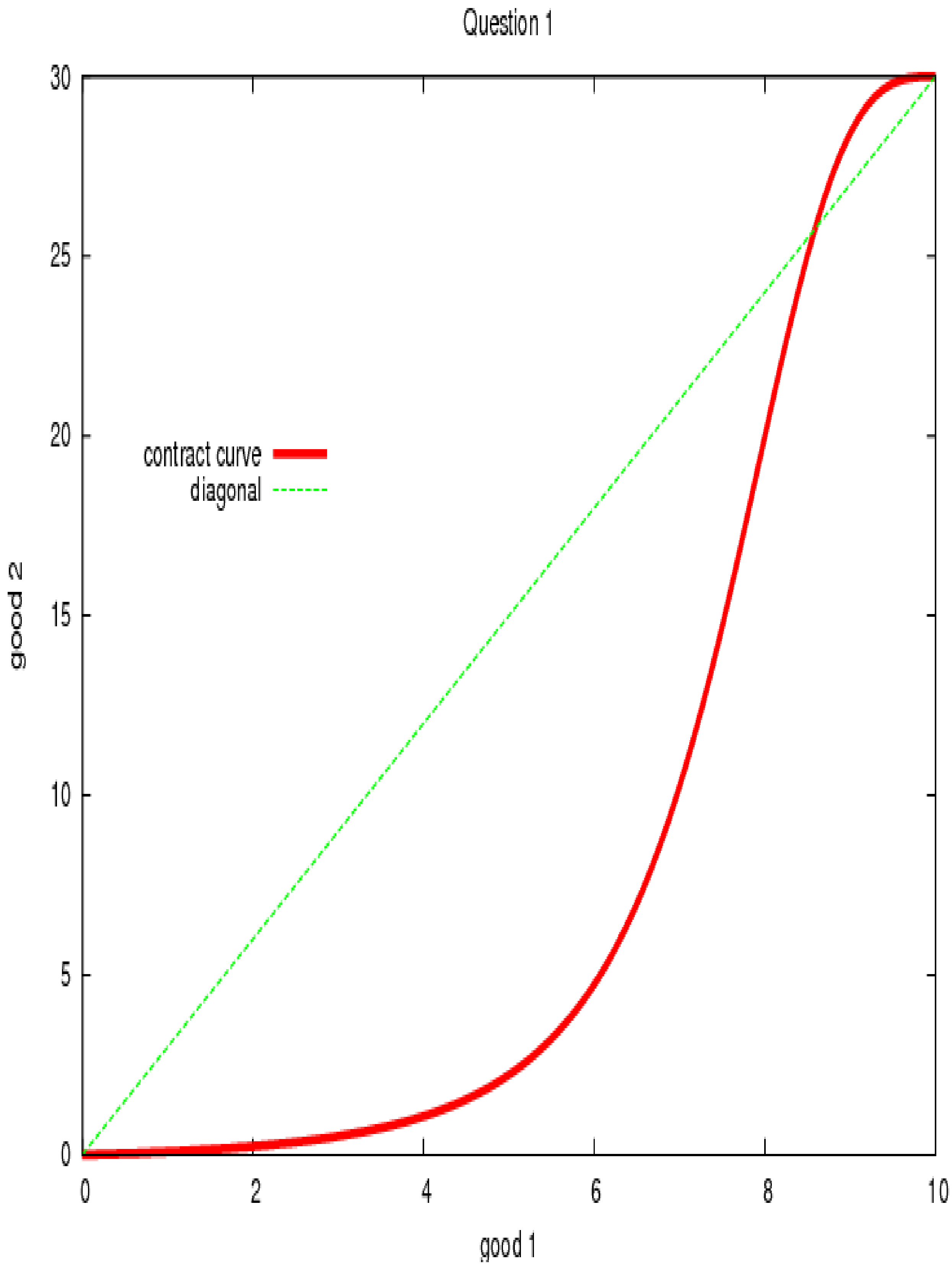
$$x_2^1 = \frac{60x_1^1}{(10 - x_1^1)^3 + 2x_1^1} \qquad (1-4)$$

If x_1^1 is measured (from the left) along the bottom of the Edgeworth box, and x_2^1 (from the bottom) along the left side, equation $(1-4)$ defines a curve, starting at the bottom–left corner $(0, 0)$, and going through the top–right corner $(10, 30)$.

The slope of the curve (from differentiation of the right side of $(1-4)$ with respect to x_1^1) is

$$\frac{\partial x_2^1}{\partial x_1^1} = 120(5 + x_1^1)\frac{(10 - x_1^1)^2}{((10 - x_1^1)^3 + 2x_1^1)^2} \qquad (1-5)$$

The contract curve slopes up : the right side of $(1-5)$ is positive for $0 < x_1^1 < 10$. But the curve is S–shaped, as in figure 1 : the slope actually is zero at the top right corner of the Edgeworth box (and close to zero at the bottom left corner).



Question 1
contract curve
diagonal
good 2
good 1
0
5
10
15
20
25
30
0
2
4
6
8
10

*Q*2. What are all the allocations in the core of a 3–person, 2–good economy, in which each person's preferences can be represented by the utility function

$$u^i(x_1^i, x_2^i) = x_1^i + 2\sqrt{x_2^i}$$

where x_j^i is person i's consumption of good j, and where the endowments e^i of the three people are $e^1 = (4, 0)$, $e^2 = (0, 4)$, $e^3 = (2, 2)$?

*A*2. Notice first that each person's marginal rate of substitution of good 1 for good 2 is

$$MRS^i = \sqrt{x_2^i} \qquad (2-1)$$

so that any coalition (of 2 people, or of all 3 people) will divide its aggregate resources efficiently if it divides the stock of good 2 equally among coalition members. (Only if $x_2^i = x_2^m$ for any members i and m of the coalition, will the two people have the same MRS.)

So, for example, if persons 1 and 3 were to form a coalition, they would have 6 units of good 1, and 2 units of good 2 to divide. The best that coalition can do for its members is to divide the 2 units of good 2 equally between the two members : 1 unit each. The efficient allocations which that coalition (of persons 1 and 3) can achieve, are any allocations $\mathbf{x}^1$,$\mathbf{x}^2$ in which $\mathbf{x}^1 = (a, 1)$ and $\mathbf{x}^2 = (b, 1)$, with $a + b = 6$.

Notice next what would be the **sum** of the utilities of the members of the coalition, if they divided the coalition's resources efficiently. In my example,

$$u^1 + u^3 = (a + 2\sqrt{1}) + (b + 2\sqrt{1}) = (a + b) + 4\sqrt{1} = 10$$

More generally, if the coalition has A units of good 1, and B units of good 2 to divide among its 2 members i and m, then it will give each member $B/2$ units of good 2, so that

$$u^i + u^m = A + 4\sqrt{B/2} \qquad (2-2)$$

and the "coalition of the whole", consisting of all 3 people, can achieve a total utility of all people of

$$u^1 + u^2 + u^3 = 6 + 6\sqrt{2} = 6(1 + \sqrt{2}) \qquad (2-3)$$

if it divides the available 6 units of good 2 equally among all 3 people.

Because of the **quasi–linear** utility function, what any coalition can do can be represented by a (constant) total utility for its members ; the "utility possibility frontier" for any coalition is linear. (Cooperative games which have this property, that the sum of the payoffs of the members of any coalition are constant, are called games with "transferable utility".)

So let U^{ijm} represent the total utility which can be achieved by all members of a coalition with people i, j and m in it. Then

$$U^1 = 4 \qquad (2-4)$$

$$U^2 = 4 \tag{2-5}$$

$$U^3 = 2(1+\sqrt{2}) \approx 4.8284 \tag{2-6}$$

$$U^{12} = 4+4\sqrt{2} = 4(1+\sqrt{2}) \approx 9.6569 \tag{2-7}$$

$$U^{13} = 10 \tag{2-8}$$

$$U^{23} = 2+4\sqrt{3} \approx 8.9282 \tag{2-9}$$

$$U^{123} = 6(1+\sqrt{2}) \approx 14.4853 \tag{2-10}$$

If an allocation is in the core, then it must be Pareto optimal. So an allocation in the core must be of the form $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\} = \{(a,2),(b,2),(c,2)\}$, with $a+b+c=6$.

An allocation must be "individually rational" if it is in the core, so each person must do at least as well as she would on her own. Since person 3's endowment is $(2,2)$, it must be true that $c \geq 2$, if the allocation is in the core, since otherwise person 3 would not join.

But what if $c > 2$? Persons 1 and 2 can form a coalition with an aggregate endowment of $(4,4)$. If $c > 2$, then $a+b < 4$. But then persons 1 and 2 could block the allocation by forming a coalition of their own, in which they each get $(a',2)$ and $(b',2)$ respectively, with $a' = a+(c-2)/2 > a$, and $b' = b+(c-2)/2 > b$.

So in any core allocation, person 3 gets $\mathbf{x}^3 = (2,2)$: any less of good 1 and she would not join the coalition, any more of good 1 and persons 1 and 2 could block the allocation.

Persons 1 and 2 must get $\mathbf{x}^1 = (a,2)$, and $\mathbf{x}^2 = (b,2)$, with $a+b=4$. How small can a be? If a is too small, person 1 will try and block the allocation by forming a coalition with person 3. [Why person 3? If person 1 and person 2 form a coalition, they **cannot** block an allocation in which $a+b=4$, since the total utility their coalition can offer its two members is $U^{12} = 4+4\sqrt{2} = a+b+4\sqrt{2}$, which is exactly what the two people are getting, in total, from the proposed allocation.]

If person 1 and 3 form a coalition to block an allocation, person 3 must get a utility of at least $2(1+\sqrt{2})$ if she joins, since that is the utility she gets in any core allocation. What does that leave for person 1? She will get

$$U^{13} - 2(1+\sqrt{2}) = 10 - 2(1+\sqrt{2}) \approx 5.1716 \tag{2-11}$$

if she forms this coalition and gets person 3 to join. So any core allocation must offer her a utility level of at least 5.1716. Her utility in the core allocation is $a+2\sqrt{2}$, so that the amount of good 1 she gets, a, must be at least a_{min}, where $a_{min}+2\sqrt{2} = 5.1716$, or

$$a_{min} = 10 - 2(1+\sqrt{2}) - 2\sqrt{2} = 8 - 4\sqrt{2} \approx 2.3431 \tag{2-12}$$

If person 1's allocation b of good 1 is too small, then he can try and form a coalition with person 3 to block the allocation. If she gives person 3 her required level of utility $2(1+\sqrt{2}$, then that leaves person 2 with a utility of

$$U^{23} - 2(1+\sqrt{2})4\sqrt{3} - 2\sqrt{2} \approx 4.1 \quad (2-13)$$

The amount of good 2 she gets, which will guarantee her this minimum level of utility, is the quantity b_{min} for which $b_{min} + 2\sqrt{2} = 4.1$, or

$$b_{min}4\sqrt{3} - 2\sqrt{2} - 2\sqrt{2} \approx 1.2713 \quad (2-14)$$

Here, if $a > a_{min}$ and if $b > b_{min}$, then neither person 1 nor person 2 will be better off leaving the coalition to go off on her or his own.

Since $a + b = 4$, then a cannot exceed $4 - b_{min}$ if the allocation is in the core. So the core allocations are all the allocations $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\} = \{(a, 2), (b, 2), (2, 2)\}$, with

$$a + b = 4 \quad (2-15)$$

$$2.3431 \leq a \leq 2.7287 \quad (2-16)$$

$$1.2713 \leq b \leq 1.6769 \quad (2-17)$$

*Q*3. In the economy described in question #1 above, suppose that person 2's endowment of the two goods is $\mathbf{e}^2 = (\alpha, 8)$. Suppose as well that person 1 chooses to consume 8 units of good 1 in the resulting competitive equilibrium.

What does α equal?

*A*3. Every competitive equilibrium must be Pareto optimal. Therefore, the competitive equilibrium must be on the contract curve. If person 1 chooses to consume 8 units of good 1 in equilibrium, then (x_1^1, x_2^1) must satisfy equation $(1-4)$ defining the Pareto optimal optimal allocations, with $x_1^1 = 8$. Plugging $x_1^1 = 8$ into equation $(1-4)$ yields

$$x_2^1 = 20 \quad (3-1)$$

In equilibrium, person 1 chooses a consumption bundle for which her MRS equals the price ratio. So

$$MRS^1 = \frac{x_2^1}{x_1^1} = \frac{20}{8} = \frac{p_1}{p_2} \quad (3-2)$$

which defines the equilibrium price ratio.

Since person 2's endowment is $(\alpha, 8)$, then person 1's endowment is $\mathbf{e}^1 = (10 - \alpha, 22)$, if the total endowments of the goods are $(10, 30)$.

Person 1's consumption bundle must be on her budget line, so that

$$8p_1 + 20p_2 = (10 - \alpha)p_1 + 22p_2 \qquad (3-3)$$

or

$$(\alpha - 2)\frac{p_1}{p_2} = 2 \qquad (3-4)$$

Plugging in for the price ratio from $(3-2)$,

$$(\alpha - 2)\frac{20}{8} = 2 \qquad (3-5)$$

implying that $\alpha = 2.8$.

*Q*4. Calculate the competitive equilibrium for the 3–person, 2–good economy described in question #2.

*A*4. The competitive equilibrium allocation is the allocation for which the sum of the people's demand for a good equals the total endowment of the good.

It is easiest to look at the excess demand for good 2. Since people's preferences are quasi–linear, each person's demand for good 2 does not depend on her income.

Since a person's optimal consumption bundle is one for which her MRS equals the price ratio, equation $(2-1)$ implies that

$$\sqrt{x_2^i} = \frac{p_1}{p_2} \qquad (4-1)$$

or

$$x_2^i = (\frac{p_1}{p_2})^2 \qquad (4-2)$$

The total demand for good 2 is $x_2^1 + x_2^2 + x_2^3$, which must equal the total endowment of the good, 6 units. Therefore

$$x_2^1 + x_2^2 + x_2^3 = 3(\frac{p_1}{p_2})^2 = 6 \qquad (4-3)$$

so that the equilibrium price ratio must be

$$\frac{p_1}{p_2} = \sqrt{2} \qquad (4-4)$$

Plugging $(4-4)$ into $(4-2)$, each person chooses to consume $x_2^i = 2$ units of good 2 in equilibrium. (This must be true : $x_2^1 = x_2^2 = x_2^3 = 2$ in any Pareto optimal allocation here, and the competitive equilibrium must be Pareto optimal.)

The value of person 1's endowment is $4p_1$, so that her equilibrium consumption bundle $(x_1^1, x_2^1) = (x_1^1, 2)$ must equal the value of her endowment

$$4p_1 = x_1^1 p_1 + 2p_2 \qquad (4-5)$$

or

$$x_1^1 = 4 - 2\frac{p_2}{p_1} = 4 - 2\frac{1}{\sqrt{2}} = 4 - \sqrt{2} \approx 2.5858 \qquad (4-6)$$

The value of person 2's endowment is $4p_2$, so that she consumes the bundle $(x_1^2, x_2^2) = (x_1^2, 2)$ such that

$$4p_2 = p_1 x_1^2 + 2p_2 \qquad (4-7)$$

or

$$x_1^2 = 2\frac{p_2}{p_1} = \sqrt{2} \approx 1.4142 \qquad (4-7)$$

Person 3 has an endowment of good 3 of 2, and chooses to consume 2 units of good 2 in equilibrium, so that being on her budget line requires her to consume $x_1^3 = e_1^3 = 2$.

Therefore, the equilibrium allocation is $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\} = \{(4-\sqrt{2}, 2), (\sqrt{2}, 2), (2, 2)\}$.

[Note that once x_1^1 is known from equation $(4-6)$, the values of x_1^2 and x_1^3 follow immediately from the facts that a competitive equilibrium allocation must be in the core, and that $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\} = \{(4-\sqrt{2}, 2), (\sqrt{2}, 2), (2, 2)\}$ is the only core allocation for which $x_1^1 = 4 - \sqrt{2}$.]

[Note as well that Walras's Law means that, in order to find the equilibrium, it is only necessary to use the market–clearing condition for one of the two markets.]

*Q*5. Find all the pure–strategy Nash equilibria in the following strategic–form two–person game.

	a	b	c	d	e	f	g
A	$(0,0)$	$(0,8)$	$(0,15)$	$(0,17.7)$	$(0,20)$	$(0,23)$	$(0,24)$
B	$(8,0)$	$(4,4)$	$(2,9)$	$(1,10.7)$	$(0,12)$	$(-2,13)$	$(-4,12)$
C	$(15,0)$	$(9,2)$	$(6,6)$	$(4.5,7.2)$	$(3,8)$	$(0,8)$	$(-3,6)$
D	$(17.7,0)$	$(10.7,1)$	$(7.2,4.5)$	$(5.5,5.5)$	$(3.7,6)$	$(0.2,5.5)$	$(-3.2,3)$
E	$(20,0)$	$(12,0)$	$(8,3)$	$(6,3.7)$	$(4,4)$	$(0,3)$	$(-4,0)$
F	$(23,0)$	$(13,-2)$	$(8,0)$	$(5.5,0.2)$	$(3,0)$	$(-2,-2)$	$(-7,-6)$
G	$(24,0)$	$(12,-4)$	$(6,-3)$	$(3,-3.2)$	$(0,-4)$	$(-6,-7)$	$(-12,-12)$

*A*5. The payoffs for this game are the profits of the two Cournot duopolists of question #4 in assignment 3, when the strategies of player 1 are the quantities $A = 0$, $B = 2$, $C = 3$, $D = 3.5$, $E = 4$, $F = 5$ and $G = 6$ (rounded to one decimal place).

And the pure–strategy Nash equilibria to this game are the three Cournot–Nash equilibria from question #4 of assignment 3 : (A, g), $G, a)$ and (E, e).

But this must be checked. The game depicted in the question here is not exactly the game played by the duopolists, since here they are restricted to choosing from a finite set of (7) pure strategies, whereas the duopolists in assignment 3 could choose any level of output, whether or not it was an integer.

Here an exhaustive scrutiny of the 49 cells in the strategic form game matrix shows that the three pairs of cells just mentioned are indeed the only Nash equilibria in pure strategies. At any other cell, either player 1 wants to move up or down from the cell, or player 2 wants to move left or right.