*Q*1. Are the preferences described below strictly monotonic? Convex? Explain briefly.

The two goods are avocado and bread. Each avocado has 2 grams of protein and 500 calories. Each piece of bread has 1 gram of protein and 100 calories. The person calculates the total number of grams of protein, and the total number of calories, in each bundle. He prefers a bundle with A avocadoes and B pieces of bread to another bundle containing a avocadoes and b pieces of bread if and only if the bundle (A, B) gives her more protein per calorie than the bundle (a, b). (If the two bundles have the same protein per calorie, then he is indifferent between them.)

*A*1. What matters to this person is the ratio of the protein in some bundle, to the number of calories. Given that each avocado has 2 grams of protein and each piece of bread has one, a bundle (a, b) contains $2a + b$ grams of protein. Given that each avocado has 500 calories and each piece of bread has 100 calories, this bundle (a, b) contains $500a + 100b$ calories.

So we can construct a utility function for these preferences :

$$U(a,b) = \frac{2a+b}{500a+100b} \qquad (1-1)$$

(Of course any monotonic increasing transformation of $U(a, b)$ will also serve as a utility function.)

But this utility function can also be written

$$f(\beta) = \frac{2+\beta}{500+100\beta} \qquad (1-2)$$

where

$$\beta \equiv \frac{b}{a}$$

is the **ratio** of pieces of bread to avocadoes in the bundle.

In other words, all that matters to this person is this ratio : how many pieces of bread per avocado. Differentiating,

$$f'(\beta) = \frac{300}{(500+100\beta)^2} > 0$$

Therefore, this person's preferences are not strictly monotonic : holding constant the number of pieces of bread, she would prefer the bundle with fewer avocadoes.

Or more directly, equation $(1-1)$ shows that $\frac{\partial U(a,b)}{\partial a} < 0$.

To check convexity of preferences, one way is to go directly to the original definition of convexity. Suppose that the bundles (a, b) and (A, B) were both at least as good as some bundle $(\bar{a}, \bar{b})$. That means that

$$\frac{b}{a} \geq \bar{\beta} \equiv \frac{\bar{b}}{\bar{a}} \qquad (1-3)$$

$$\frac{B}{A} \geq \bar{\beta} \equiv \frac{\bar{b}}{\bar{a}} \qquad (1-4)$$

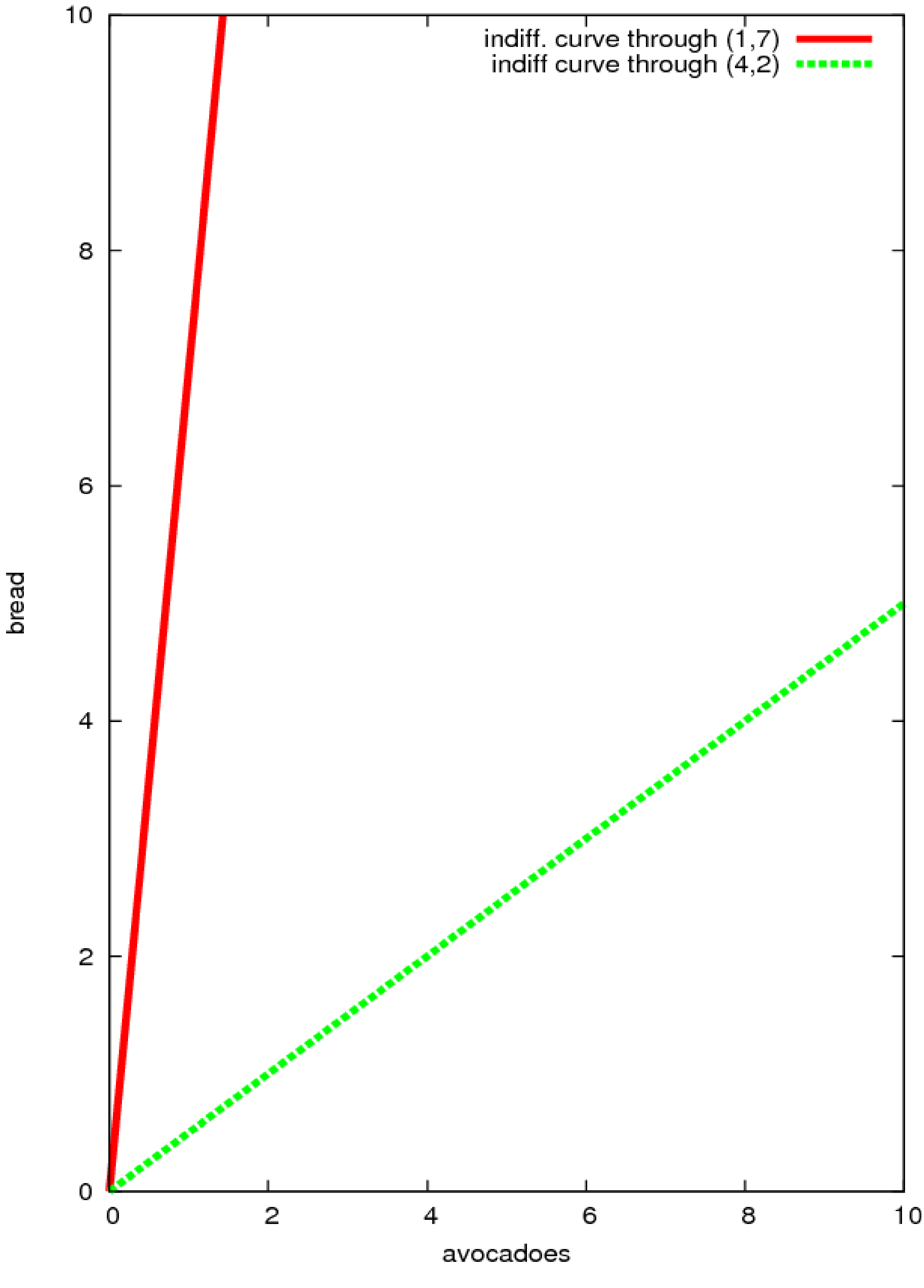

Figure 1 : indifference curves for question 1 (“good” direction is up)

Now take some convex combination $(ta + (1-t)A, tb + (1-t)B)$ of the two bundles (a, b) and (A, B). What is the ratio of bread to avocadoes in this new bundle? : $\frac{tb+(1-t)B}{ta+(1-t)A}$ Now

$$\frac{tb + (1-t)B}{ta + (1-t)A} \geq \beta \quad \text{if and only if} \quad tb + (1-t)B \geq ta\beta + (1-t)A\beta \qquad (1-5)$$

But $(1-3)$ and $(1-4)$ imply that $tb \geq ta\beta$ and $(1-t)B \geq (1-t)A\beta$, so that the inequality in $(1-5)$ must hold.

Therefore, this person's preferences are convex (but not strictly convex).

Alternatively, equation $(1-2)$ implies that the "at least as good as" set for some bundle (a, b) consists of all bundles on a line from the origin going through the point (a, b), and all bundles below and to the right of this line (when we graph a on the horizontal and b on the vertical). Figure 1 shows two indifference curves. In that figure, the "at least as good as" set for the bundle $(4, 2)$ is all the bundles in the wedge–shaped area above and on the green line. That wedge–shaped area is a convex set.

Summarizing, these preferences are not strictly monotonic, but they are convex.

*Q*2. Are the preferences represented by the utility function below strictly monotonic? Convex? Explain briefly.

$$u(x_1, x_2) = \frac{x_1 x_2}{x_1 + x_2} \quad \text{if} \quad (x_1, x_2) \neq (0, 0)$$
$$u(0, 0) = 0$$

*A*2. These preferences are strictly monotonic. Partial differentiation of the utility function shows that

$$\frac{\partial u}{\partial x_1} = [\frac{x_2}{x_1 + x_2}]^2 > 0$$
$$\frac{\partial u}{\partial x_2} = [\frac{x_1}{x_1 + x_2}]^2 > 0$$

(Also, since $u(x_1, x_2) > 0$ whenever $(x_1, x_2) >> (0, 0)$, utility is strictly monotonic near $\mathbf{x} = (\mathbf{0}, \mathbf{0})$.)

The easiest way to check convexity is to take an increasing monotonic transformation of the utility function. Let

$$F(u) = A - \frac{1}{u}$$

which is a strictly increasing function. Then

$$U(x_1, x_2) \equiv F(u(x_1, x_2)) = A - \frac{x_1 + x_2}{x_1 x_2} = A - \frac{1}{x_1} - \frac{1}{x_2}$$

The function $U(x_1, x_2)$ is a concave function : $U_{ii} < 0$ and $U_{ij} = 0$ for $i \neq j$.

Therefore, these preferences are strictly monotonic, and (strictly) convex.

*Q*3. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = \ln(x_1) + 2\sqrt{x_2 x_3}$$

*A*3. If the person maximizes the utility function $u(x_1, x_2, x_3)$ subject to her budget constraint $\mathbf{p} \cdot \mathbf{x} \leq m$, then the first–order conditions for the maximization are

$$u_1 = \frac{1}{x_1} = \lambda p_1 \tag{3 – 1}$$

$$u_2 = \sqrt{\frac{x_3}{x_2}} = \lambda p_2 \tag{3 – 2}$$

$$u_3 = \sqrt{\frac{x_2}{x_3}} = \lambda p_3 \tag{3 – 3}$$

where λ is the Lagrange multiplier on the budget constraint.

Combining equations (3 – 2) and (3 – 3),

$$\frac{x_2}{x_3} = \frac{p_3}{p_2} \tag{3 – 4}$$

Equation (3 – 1) can be written

$$\lambda = \frac{1}{p_1 x_1} \tag{3 – 5}$$

Plugging (3 – 4) and (3 – 5) back into (3 – 2)

$$\sqrt{\frac{p_2}{p_3}} = \frac{p_2}{p_1 x_1} \qquad 3 – 6)$$

Equation (3 – 6) now expresses the quantity demanded of good 1 as a function of prices of all the goods. It can be re–arranged into

$$x_1 = \frac{\sqrt{p_2 p_3}}{p_1} \tag{3 – 7}$$

which is the Marshallian demand function for good 1.

Now the person's budget constraint can be written

$$p_2 x_2 + p_3 x_3 = m - p_1 x_1 \tag{3 – 8}$$

Using (3 – 4) and (3 – 7), this budget constraint becomes

$$2 p_2 x_2 = m - \sqrt{p_2 p_3} \tag{3 – 9}$$

or

$$x_2 = \frac{1}{2}(\frac{m}{p_2} - \sqrt{\frac{p_3}{p_2}}) \qquad (3-10)$$

which is the Marshallian demand function for good 2. Plugging in equation $(3-4)$ again,

$$x_3 = \frac{1}{2}(\frac{m}{p_3} - \sqrt{\frac{p_2}{p_3}}) \qquad (3-11)$$

Summarizing, the Marshallian demand functions for this consumer are

$$x_1(\mathbf{p}, m) = \frac{\sqrt{p_2 p_3}}{p_1}$$

$$x_2(\mathbf{p}, m) = \frac{1}{2}(\frac{m}{p_2} - \sqrt{\frac{p_3}{p_2}})$$

$$x_3(\mathbf{p}, m) = \frac{1}{2}(\frac{m}{p_3} - \sqrt{\frac{p_2}{p_3}})$$

It can be checked that these demand functions are each homogeneous of degree 0 in prices and income together, and that they exactly exhaust the person's budget constraint.

The requirement $\sqrt{p_2 p_3} < m$ is necessary to ensure that $p_1 x_1 < m$, so that quantities demanded of goods 2 and 3 are non–negative. If prices of goods 2 and 3 were so high that $\sqrt{p_2 p_3} > m$, then the person would choose not to consume any of goods 2 and 3, and would spend all her money on good 1.

*Q*4. Calculate a person's Marshallian demand functions, and her expenditure function, if her direct utility function is

$$u(x_1, x_2, x_3) = 2(\sqrt{x_1 x_2} + \sqrt{x_1 x_3})$$

*A*4. The first–order conditions for maximization of the person's utility, subject to the budget constraint, are

$$\sqrt{\frac{x_2}{x_1}} + \sqrt{\frac{x_3}{x_1}} = \lambda p_1 \qquad (4-1)$$

$$\sqrt{\frac{x_1}{x_2}} = \lambda p_2 \qquad (4-2)$$

$$\sqrt{\frac{x_1}{x_3}} = \lambda p_3 \qquad (4-3)$$

where λ is the Lagrange multiplier on the person's budget constraint.

Substituting from $(4-3)$ and $(4-2)$ into equation $(4-1)$,

$$\frac{1}{\lambda p_2} + \frac{1}{\lambda p_3} = \lambda p_1 \qquad (4-4)$$

or

$$\lambda^2 = \frac{p_2 + p_3}{p_1 + p_2 + p_3} \tag{4 – 5}$$

Since equations $(4-2)$ and $(4-3)$ imply that

$$x_2 = \frac{1}{\lambda^2 p_2^2} x_1 \tag{4 – 6}$$

$$x_3 = \frac{1}{\lambda^2 p_3^2} x_1 \tag{4 – 7}$$

the person's budget constraint $\mathbf{p} \cdot \mathbf{x} = m$ can be written

$$(p_1 + \frac{1}{\lambda^2 p_2^2} + \frac{1}{\lambda^2 p_3^2}) x_1 = m \tag{4 – 8}$$

Substituting for λ^2 from equation $(4-5)$ means that equation $(4-8)$ expresses the quantity demanded of good 1 as a function of income and prices :

$$(p_1 + \frac{p_1 p_3}{p_2 + p_3} + \frac{p_1 p_2}{p_2 + p_3}) x_1 = m \tag{4 – 9}$$

or

$$x_1(\mathbf{p}, m) = \frac{m}{2p_1} \tag{4 – 10}$$

Substitution of $(4-10)$ (and $(4-5)$) into $(4-6)$ and $(4-7)$ yields the Marshallian demand functions for goods 2 and 3,

$$x_2(\mathbf{p}, m) = \frac{p_3}{p_2(p_2 + p_3)} \frac{m}{2} \tag{4 – 11}$$

$$x_3(\mathbf{p}, m) = \frac{p_2}{p_3(p_2 + p_3)} \frac{m}{2} \tag{4 – 12}$$

Since $(4-10)$ – $(4-12)$ imply that

$$\sqrt{x_1 x_2} = \frac{m}{2} \frac{1}{\sqrt{p_1 p_2 p_3}} \frac{p_3}{\sqrt{p_2 + p_3}} \tag{4 – 13}$$

$$\sqrt{x_1 x_3} = \frac{m}{2} \frac{1}{\sqrt{p_1 p_2 p_3}} \frac{p_2}{\sqrt{p_2 + p_3}} \tag{4 – 14}$$

therefore the indirect utility function can be written

$$v(\mathbf{p}, m) = \frac{\sqrt{p_2 + p_3}}{\sqrt{p_1 p_2 p_3}} m \tag{4 – 15}$$

Since $v[\mathbf{p}, e(\mathbf{p}, u)] = u$, equation $(4-16)$ implies that the person's expenditure function is

$$e(\mathbf{p}, u) = \frac{\sqrt{p_1 p_2 p_3}}{\sqrt{p_2 + p_3}} u \tag{4 – 16}$$

Alternatively, the expenditure function can be obtained directly, by minimization of $\mathbf{p} \cdot \mathbf{x}$ with respect to $\mathbf{x}$ subject to the "minimum utility" constraint $2(\sqrt{x_1 x_2} + \sqrt{x_1 x_3}) = u$. The first–order conditions for this minimization are

$$\mu \sqrt{\frac{x_2}{x_1}} + \sqrt{\frac{x_3}{x_1}} = p_1 \qquad (4-17)$$

$$\mu \sqrt{\frac{x_1}{x_2}} = p_2 \qquad (4-18)$$

$$\mu \sqrt{\frac{x_1}{x_3}} = p_3 \qquad (4-19)$$

Equations $(4-18)$ and $(4-19)$ imply that

$$\mu^2 = \frac{p_1 p_2 p_3}{p_2 + p_3} \qquad (4-20)$$

Since $(4-18)$ and $(4-19)$ imply that

$$x_2 = (\frac{\mu}{p_2})^2 x_1 \qquad (4-21)$$

$$x_3 = (\frac{\mu}{p_3})^2 x_1 \qquad (4-22)$$

therefore the minimum utility constraint $2(\sqrt{x_1 x_2} + \sqrt{x_1 x_3}) = u$ can be written

$$u = 2\mu(\frac{1}{p_2} + \frac{1}{p_3}) x_1 \qquad (4-23)$$

Using $(4-20)$ to solve for μ, equation $(4-23)$ implies that the Hicksian demand function for good 1 is

$$x_1^H(\mathbf{p}, u) = \frac{1}{2} \frac{\sqrt{p_2 p_3}}{\sqrt{p_1}\sqrt{p_2 + p_3}} u \qquad (4-24)$$

Since $(4-18)$–$(4--20)$ imply that

$$x_2^H(\mathbf{p}, u) = \frac{p_1 p_3}{p_2(p_2 + p_3)} x_1^H(\mathbf{p}, u) \qquad (4-25)$$

$$x_3^H(\mathbf{p}, u) = \frac{p_1 p_2}{p_3(p_2 + p_3)} x_1^H(\mathbf{p}, u) \qquad (4-26)$$

therefore

$$e(\mathbf{p}, u) = p_1 x_1^H(\mathbf{p}, u) + p_2 x_2^H(\mathbf{p}, u) + p_3 x_3^H(\mathbf{p}, u) = 2 p_1 x_1^H(\mathbf{p}, u) \qquad (4-27)$$

so that equation $(4-24)$ implies that

$$e(\mathbf{p}, u) = \frac{\sqrt{p_1 p_2 p_3}}{\sqrt{p_2 + p_3}} u$$

which is just equation $(4-16)$.

*Q*5. Calculate the expenditure function for a person whose direct utility function is

$$u(x_1, x_2) = 10 - \frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{x_2}}$$

*A*5. Solving the minimization of $p_1x_1 + p_2x_2$ subject to the minimum utility constraint $10 - \frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{x_2}} = u$ yields first–order conditions

$$\frac{\mu}{2}x_1^{-3/2} = p_1 \tag{5 – 1}$$

$$\frac{\mu}{2}x_2^{-3/2} = p_2 \tag{5 – 2}$$

where μ is the Lagrange multiplier on the minimum utility constraint. These equations imply that

$$x_2^H(\mathbf{p}, u) = (\frac{p_1}{p_2})^{2/3}x_1^H(\mathbf{p}, u) \tag{5 – 3}$$

or

$$\frac{1}{\sqrt{x_2}} = (\frac{p_2}{p_1})^{1/3}\frac{1}{\sqrt{x_1}} \tag{5 – 4}$$

Substituting (5 – 4) back into the minimum utility constraint,

$$u = 10 - \frac{1}{\sqrt{x_1^H(\mathbf{p}, u)}}(1 + (\frac{p_2}{p_1})^{1/3}) \tag{5 – 5}$$

so that

$$\frac{1}{10 - u} = \sqrt{x_1^H\mathbf{p}, u)}\frac{(p_1)^{1/3}}{(p_1)^{1/3} + (p_2)^{2/3}} \tag{5 – 6}$$

That means that the Hicksian demand for good 1 is

$$x_1^H(\mathbf{p}, u) = (10 - u)^{-2}\frac{[(p_1)^{1/3} + (p_2)^{1/3}]^2}{(p_1)^{2/3}} \tag{5 – 7}$$

Since $e(\mathbf{p}, u) = p_1x_1^H(\mathbf{p}, u) + p_2x_2^H(\mathbf{p}, u)$, equations (5 – 4) and (5 – 7) imply that the person's expenditure function is

$$e(\mathbf{p}, u) = (10 - u)^{-2}[(p_1)^{1/3} + (p_2)^{1/3}]^3 \tag{5 – 8}$$

This could also be solved using the (textbook) formulae for CES preferences, since the preferences here are CES, with $\rho = -1/2$.