*Q*1. Find the profit function, supply function, and unconditional input demand functions for a firm with a production function

$$f(x_1, x_2) = 2\sqrt{x_1} + \ln(x_2 + 1)$$

(do not assume that $w_2$ must be less than $p$, where $w_2$ is the price of input 2, and $p$ is the price of output)

*A*1. The profit function can be defined as the maximum with respect to the output level $y$ of $py - C(\mathbf{w}, y)$. But here it is rather difficult to obtain the cost function $C(\mathbf{w}, y)$ in closed form.

An alternative way of finding the competitive firm's profit function is to maximize $pf(x_1, x_2) - w_1x_1 - w_2x_2$ with respect to $x_1$ and $x_2$.

The first–order conditions for this latter maximization problem are

$$p\frac{\partial f}{\partial x_1} = \frac{p}{\sqrt{x_1}} = w_1 \tag{1 − 1}$$

$$p\frac{\partial f}{\partial x_2} = \frac{p}{x_2 + 1} = w_2 \tag{1 − 2}$$

Equation $(1 - 1)$ and $(1 - 2)$ can be solved directly for $x_1$ and $x_2$ respectively :

$$x_1 = (\frac{p}{w_1})^2 \tag{1 − 3}$$

$$x_2 = \frac{p}{w_2} - 1 \tag{1 − 4}$$

Equations $(1 - 3)$ and $(1 - 4)$ are the firm's unconditional input demand functions $\mathbf{x}^u(p, \mathbf{w})$. But they only apply if $p \geq w_2$. If $p < w_2$, equation $(1 - 4)$ implies a negative value for $x_2$, which cannot be correct. When $p < w_2$, then the firm chooses not to use any of input 2. This latter case will be considered below.

But if $p \geq w_2$, then $(1 - 3)$ and $(1 - 4)$ can be substituted into the production function to calculate what is the firm's profit–maximizing output level.

$$y = f(x_1, x_2) = 2\sqrt{[\frac{p}{w_1}]^2} + \ln(\frac{p}{w_2}) = 2\frac{p}{w_1} + \ln p - \ln w_2 \tag{1 − 5}$$

Equation $(1 - 5)$ defines the firm's supply function $y(p, \mathbf{w})$. Since profit is $py - w_1x_1 - w_2x_2$, substitution from $(1 - 3)$ – $(1 - 5)$ implies that

$$\pi(p, \mathbf{w}) = 2\frac{p^2}{w_1} + p\ln p - p\ln w_2 - \frac{p^2}{w_1} - p + w_2 = \frac{p^2}{w_1} + p\ln p - p\ln w_2 - p + w_2 \tag{1 − 6}$$

It can be checked that Hotelling's Lemma does hold here : if expression $(1 - 6)$ is differentiated, then the derivative with respect to $p$ is expression $(1 - 5)$, the derivative with respect to $w_1$ is the negative of expression $(1-3)$ and the derivative with respect to $w_2$ is the negative of expression $(1-4)$.

When $w_2 > p$, the firm does not use input 2. In this case, then,

$$y = 2\sqrt{x_1} \tag{1-7}$$

The value of $x_1$ which maximizes $2p\sqrt{x_1} - w_1 x_1$ is

$$x_1 = (\frac{p}{w_1})^2 \tag{1-8}$$

just as in the case in which $x_2 > 0$. But now

$$y = 2\frac{p}{w_1} \tag{1-9}$$

and

$$\pi(p, \mathbf{w}) = \frac{p^2}{w_1} \tag{1-10}$$

Again Hotelling's Lemma is satisfied by expression $(1-10)$, when $x_2 = 0$.

*Q*2. What is the equation of the supply curve of a firm which has a long–run total cost function with the equation

$$TC(q) = 6q^2 - 36q + 216\frac{q}{q+1}$$

where $q$ is the quantity of output produced by the firm?

(You do not need to solve for the equation of the supply curve in closed form, just the relation between price and output supplied).

*A*2. Given the total cost function, since $AC = TC/q$ and $MC = TC'$, therefore

$$AC(q) = 6q - 36 + \frac{216}{q+1} \tag{2-1}$$

$$MC(q) = 12q - 36 + \frac{216}{(q+1)^2} \tag{2-2}$$

Therefore

$$AC'(q) = 6 - \frac{216}{(q+1)^2} \tag{2-3}$$

Expression $(2-3)$ shows that the average cost curve is $U$–shaped : $AC' = -210 < 0$ at $q = 0$ and $AC' \rightarrow 6 > 0$ as $q \rightarrow \infty$. The bottom of the average cost curve is the value of output $q$ for which expression $(2-3)$ equals zero. That expression equals zero whenever

$$(q+1)^2 = \frac{216}{6} = 36 \tag{2-4}$$

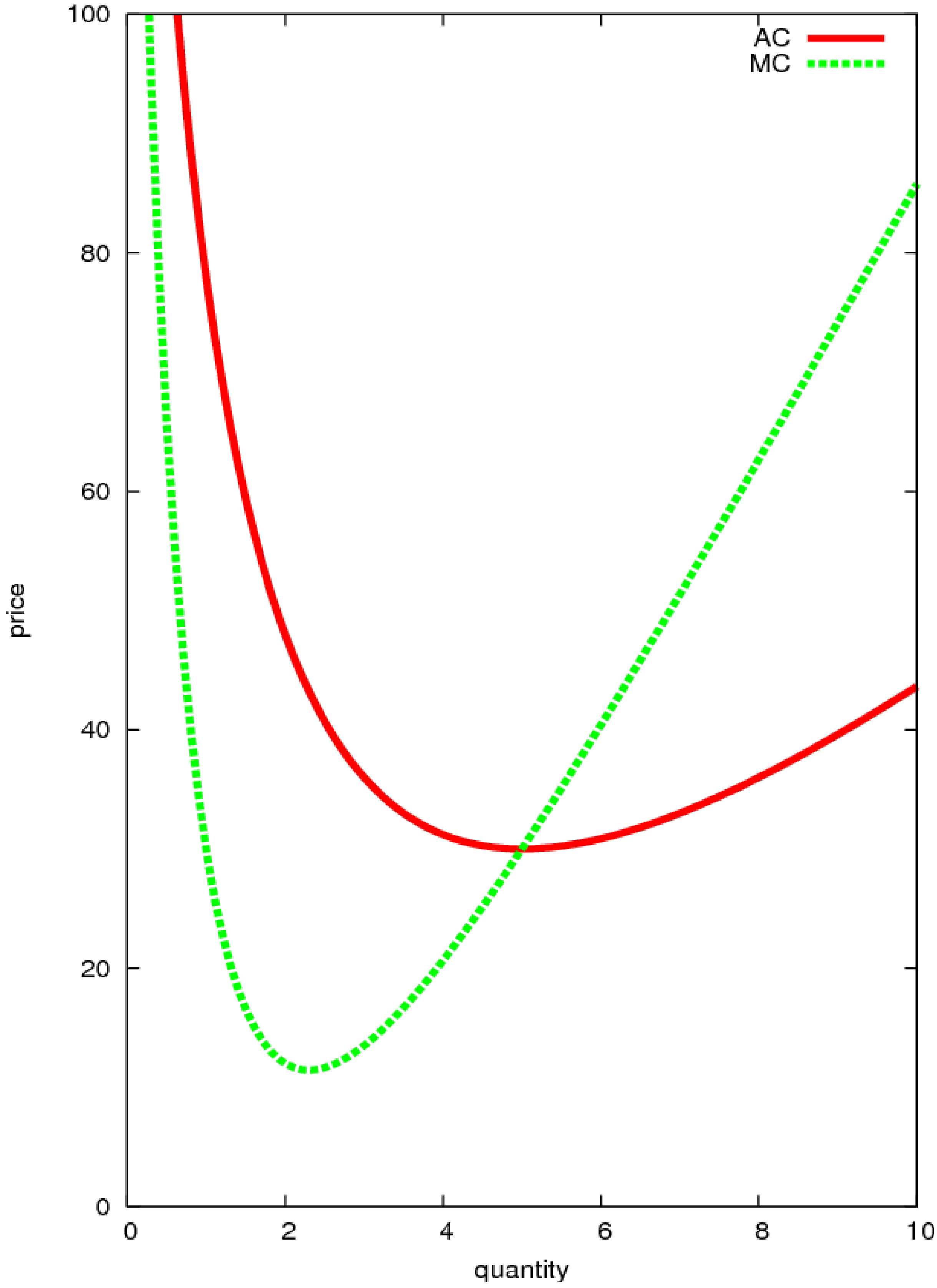




Figure : the supply curve is the MC curve above its intersection with the AC curve

The solution to equation $(2-4)$ is $q = 5$. So the bottom of the average cost curve occurs at an output level of $q = 5$.

Checking, at $q = 5$, expressions $(2-1)$ and $(2-2)$ show that

$$AC(5) = 6(5) - 36 + \frac{216}{6} = 30 \qquad (2-5)$$

$$MC(5) = 12(5) - 36 + \frac{216}{(6)^2} = 30 \qquad (2-6)$$

Therefore, the firm chooses to supply no output at all if the price is less than 30, the minimum possible level of average cost. At $p = 30$, the firm is willing to supply either an output of 0, or an output of 5. For $p > 5$, the firm's (positive) level of output is determined by the firm's marginal cost equalling the price. From equation $(2-2)$, therefore, if $p > 5$, then the firm's supply of output is the solution to

$$p = 12q - 36 + \frac{216}{(q+1)^2} \qquad (2-7)$$

Expression $(2-7)$ is a cubic equation in $q$, so is rather difficult to solve in closed form. But the accompanying figure illustrates the supply curve. In the figure, the supply curve is the vertical axis, up to a height of 30 (which is the minimum average cost) ; it then jumps, to the point $(5, 30)$, and the supply curve (for $p > 30$) is the dotted green MC curve, above its intersection with the AC curve.

*Q*3. Derive the aggregate supply curve for apples for the following imaginary area.

The area consists of the land on both sides of a river. On each side of the river, all the land within 1 km of the river can be used for agriculture. The land can be used for grape–growing or for apple–growing. The profit (net of all expenses) from growing grapes on land anywhere in the valley is 100 dollars per square kilometre of land.

If the land is used for apples, the apples must be shipped down the river to a port at the mouth of a river. It costs \$1 per km to transport a tonne of apples down the river. It costs \$10 to transport a tonne of apples to the (distant) market. (So the cost of transporting 1 tonne of apples to market will be $10 + z$, if the farm is $z$ kilometres upriver.)

Apples are grown using labour as the only input. The number of tonnes of apples which can be produced (per square kilometre of land) is

$$q = 2\sqrt{L}$$

where $L$ is the amount of labour used (per square kilometre). The wage rate of labour is \$1 per hour ; this wage rate is the same everywhere in the valley.

There are many small farms along the valley, each owned by a different farmer who can choose to use the land either for apple production or for grape production.

*A*3. Consider a farm at distance $z$ from the mouth of the river. If apples are being grown on the farm, then the farm will get a net price of $p - 10 - z$ per tonne of apples, after the costs of transporting the apples to market are paid.

How much labour would such a farm which to employ per square kilometre? It wishes to maximize the profit per square kilometre. Given the production technology, and given the wage rate of \$1 per hour, that profit is

$$2(p - 10 - z)\sqrt{L} - L \tag{3 − 1}$$

per square kilometre, if $L$ hours of labour are employed (per square kilometre). Maximizing expression $(3 - 1)$ with respect to $L$ implies picking a labour input such that

$$\frac{p - 10 - z}{\sqrt{L}} = 1 \tag{3 − 2}$$

or

$$L = (p - 10 - z)^2 \tag{3 − 3}$$

Farms further from the market employ less labour per square kilometre, because the net value of the apples produced by the labour is lower.

So, if the farmers choose the profit–maximizing quantity of labour (satisfying equation $(3-3)$), then their profit per square kilometre will be

$$2(p - 10 - z)\sqrt{L} - L = (p - 10 - z)^2 \tag{3 − 4}$$

But the farmers will only choose to grow apples if the profit from apples is greater than the profit from grapes. Expression $(3 - 4)$ shows the profit per square kilometre from apple growing at a farm $z$ kilometres from the mouth of the river. The farmer will grow apples on the land only if that profit is at least as large as the profit from grape growing, \$100 per square kilometre. So the farmer will choose to supply apples only if $(p - 10 - z)^2 \geq 100$, which is equivalent to

$$z \leq D(p) \equiv p - 20 \tag{3 − 5}$$

where $D(p)$ defines the dividing line between apple farms and grape farms.

If $p < 20$, then no farmer will choose to grow apples ; grapes are more profitable everywhere in the valley. If $p > 20$, then all farmers within $D(p)$ kilometres or less of the mouth of the river will choose to grow apples.

What then is the aggregate supply of apples from the valley? Since the farms are on both sides of the river, and are each 1 kilometre wide, then the aggregate supply of apples is

$$S(p) = 2\int_0^{D(p)} s(p, z)dz \tag{3 − 6}$$

where $s(p, z)$ is the supply of apples per square kilometre from farms at a distance $z$ from the mouth of the river, if the price is $p$.

From the production function and equation $(3-3)$, the supply of apples per square kilometres is

$$s(p,z) = 2(p-10-z) \qquad (3-7)$$

at distance $z$. (Note that Hotelling's Lemma applies : $(3-7)$ is the derivative of $(3-4)$ with respect to $p$.) From equations $(3-5)$ and $(3-7)$,

$$S(p) = 4\int_0^{p-20}(p-10-z)dz \qquad (3-8)$$

is the aggregate supply of apples from the valley when $p \geq 20$. Solving the integral

$$S(p) = 2p(p-20) \qquad (3-9)$$

*Q*4. What is the Cournot–Nash equilibrium in a duopoly in which the inverse demand function for the homogeneous output of the 2 firms is

$$p = a - Q$$

where $Q$ is aggregate output, $p$ the market price and $a$ a positive constant, if the total cost of production of firm $i$ is

$$C_i(q_i) = c_i q_i$$

where $c_1$ and $c_2$ are positive constants (not necessarily equal to each other) and $q_i$ is the output of firm $i$?

*A*4. As a Cournot duopolist, firm 1 chooses its output level $q_1$ to maximize its profit

$$(p-c_1)q_1 = (a-q_1-q_2-c_1)q_1 \qquad (4-1)$$

with respect to its own output $q_1$, taking the output level $q_2$ of the other firm as given. Solving this maximization with respect to $q_1$, the first–order condition for firm 1 is

$$(a-q_2-c_1) = 2q_1 \qquad (4-2)$$

or

$$q_1 = \frac{a-c_1}{2} - \frac{q_2}{2} \qquad (4-3)$$

which defines the **reaction function** of firm 1 to its rival's choice of output $q_2$. Similarly, firm 2 chooses its own output to maximize its profit, given $q_1$, implying a reaction function of

$$q_2 = \frac{a-c_2}{2} - \frac{q_1}{2} \qquad (4-4)$$

Equations $(4-3)$ and $(4-4)$ are two equations in 2 unknowns. In matrix form, they can be written

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{a-c_1}{2} \\ \frac{a-c_2}{2} \end{pmatrix} \qquad (4-5)$$

Solving $(4-5)$ (using Cramér's Rule), the quantities produced by the two firms in Cournot–Nash equilibrium are

$$q_1 = \frac{a - 2c_1 + c_2}{3} \qquad (4-6)$$

$$q_2 = \frac{a + c_1 - 2c_2}{3} \qquad (4-7)$$

When the two firms have the same unit costs, $c_1 = c_2$, equations $(4-6)$ and $(4-7)$ become equation $(4-15)$ in Jehle and Reny (with $J = 2$).

Equations $(4-6)$ and $(4-7)$ indicates that $(i)$ the lower–cost firm produces more output in equilibrium ; $(ii)$ an increase in a firm's unit cost will cause its rival to increase its output in equilibrium.

These equations make sense only if they both imply positive levels of production. If cost differences are large enough, the higher–cost firm may shut down. If, for example, $c_1 > c_2$, then equation $(4-6)$ implies that $q_1 \geq 0$ only if $c_1 < (a + c_2)/2$ ; otherwise firm 1 produces nothing, and firm 2 produces the monopoly output, which here is $(a - c)/2$.

*Q*5. Suppose the unit costs of the 2 duopolists in question #4 above were determined randomly. Firm $i$'s unit cost $c_i$ was $c_H$ with probability 0.5 and $c_L$ with probability 0.5, where $c_H > c_L$. Each firm's cost was determined by an independent draw from the same distribution. The firms choose their output levels after learning both their own cost draw, and the cost draw of the other firm. (That is, they each know $c_1$ and $c_2$ when they make their output decisions.)

Let $\bar{c} \equiv (0.5)(c_L + c_H)$ be the expected value of each firm's unit costs.

Each firm is risk–neutral, and wishes to maximize the expected value of its equilibrium profit.

Would firm 1 rather have a certain unit cost of $\bar{c}$, or the random draw? Would firm 1 rather have its rival (firm 2) with a certain cost $\bar{c}$, or a random draw? Explain your answer.

*A*5. The answer to question 4 above can be used here to calculate the profit a firm earns, if it has unit cost $c_i$, and if its rival has unit cost $c_j$.

Given $(4-6)$ and $(4-7)$, total output $Q \equiv q_1 + q_2$ of the duopoly is

$$Q = \frac{2a - c_1 - c_2}{3} \qquad (5-1)$$

in equilibrium if firms have unit costs $c_1$ and $c_2$, and if both firms produce positive levels of output.

Therefore the equilibrium price is

$$p = a - Q = \frac{a + c_1 + c_2}{3} \qquad (5-2)$$

(Not surprisingly, the equilibrium price is increasing in each firm's unit cost.) From $(4-6)$ and $(5-1)$, firm 1's profit in Cournot–Nash equilibrium is

$$\pi_1 = (p - c_1)q_1 = (\frac{a - 2c_1 + c_2}{3})^2 \qquad (5-3)$$

and similarly

$$\pi_2 = (\frac{a + c_1 - 2c_1}{3})^2 \qquad (5-4)$$

when the two firms' output levels are positive. Equation $(5-3)$ indicates that a firm's profit is a **convex** function of the firms' costs. The matrix of second derivatives of $\pi_1$ with respect to $c_1$ and $c_2$ is

$$H(c_1, c_2) = \frac{1}{9}\begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$$

which is a positive definite matrix.

So each firm's profit is convex in the unit costs. That means that replacing the random draw $g \equiv (0.5 \circ c_H, 0.5 \circ c_L)$ for either firm's costs with the certain cost $\bar{c} = Eg = 0.5c_H + 0.5c_L$ will lower firm 1's expected profits. If the firm's owner is risk neutral, then she will prefer strictly to have each firm's cost parameter varying, than to have it constant at its expected value.

(This convexity of one firm's profit in the other firm's cost parameter holds only if the other firm is producing a positive level of output. If $c_2 > (a + c_1)/2$, then further increases in $c_2$ do not help firm 1 ; firm 2 is already shut down. So if $a = 12$ and $c_1 = 6$, then firm 1 would rather face a rival with a certain cost of 9, than a rival whose cost was 6 with probability 0.5, and 12 with probability 0.5. A cost of 9 already guarantees that firm 2 shuts down in this case, so that increasing its rival's cost to 12 does firm 1 no good. So if $a = 12$, $c_L = 6$ and $c_H = 12$, and if firm 1's cost were the random draw $g$, then it would rather face a rival with a certain cost of $\bar{c} = 9$ than a rival which also faced the random draw $g$ for its costs.)