GS/ECON 5010   Answers to Assignment 1   October 2010

Q1. Are the preferences described below strictly monotonic? Convex? Explain briefly.

The person consumes food and clothing. In comparing two bundles, A and B, she first checks whether either bundle has more than 5 units of food. If either bundle has 5 or fewer units of food, then she prefers strictly the bundle containing more food. If both bundles have 5 or fewer units of food, then she prefers strictly the bundle containing more food. If both bundles have the same quantity of food, and if that quantity no greater than 5 units, then she is indifferent between the bundles. But if both bundles have more than 5 units of food, then she prefers strictly the bundle for which the number of units of food, plus the number of units of clothing is highest. Finally, if both bundles have more than 5 units of food, and if the number of units of food, added to the number of units of clothing, is the same for both bundles, then she is indifferent between them.

A1. It perhaps is best to consider the "at least as good as" sets, for particular bundles.

Start with a bundle containing 5 or fewer units of food. Take such a bundle, for example, the bundle $(3, 10)$ [where the first number is the quantity of food, and the second the quantity of clothing]. What bundles are at least as good as the bundle $(3, 10)$? Since the quantity of food, 3, is less than 5, some other bundle (F, C) will be at least as good as the bundle $(3, 10)$ if — and only if — the second bundle has a quantity of food F which is 3 or more.

So the "at least as good as" set for $(3, 10)$ consists of all bundles (F, C) for which $F \geq 3$. Graphically, that's a vertical line through $(3, 10)$, and everything to the right of that line [if the quantity F of food is graphed on the horizontal axis]. That's a convex set.

Now consider a bundle in which the quantity of food is greater than 5, and consider its "at least as good as" set. So take a bundle such as $(7, 8)$. What bundles are at least as good? Any bundle (F, C) for which $F + C \geq 15$ will be weakly preferred to $(7, 8)$ — provided that $F > 5$. And that's it! If $F < 5$, then the person prefers (strictly) the bundle $(7, 8)$ to (F, C), no matter how big $F + C$ is. And if $F + C < 15$, then the person prefers (strictly) $(7, 8)$ to (F, C).

So the "at least as good as" for $(7, 8)$ is everything on or to the right of the curve $F + C = 15$, which is also to the right of the vertical line $F = 5$. The "at least as good as" set does not include the vertical line $F = 5$ (at or above its intersection at $(5, 10)$ with the curve $F + C = 15$). In this case, the "at least as good as" set is still convex. [It's not closed. So preferences here are not continuous. But the question didn't actually ask about continuity of preferences.]

Are preferences strictly monotonic? The person always is made better off by more food. And she is never made worse off by more clothing (even though more clothing does her no good if the quantity of food is less than 5).

So preferences are strictly monotonic here.

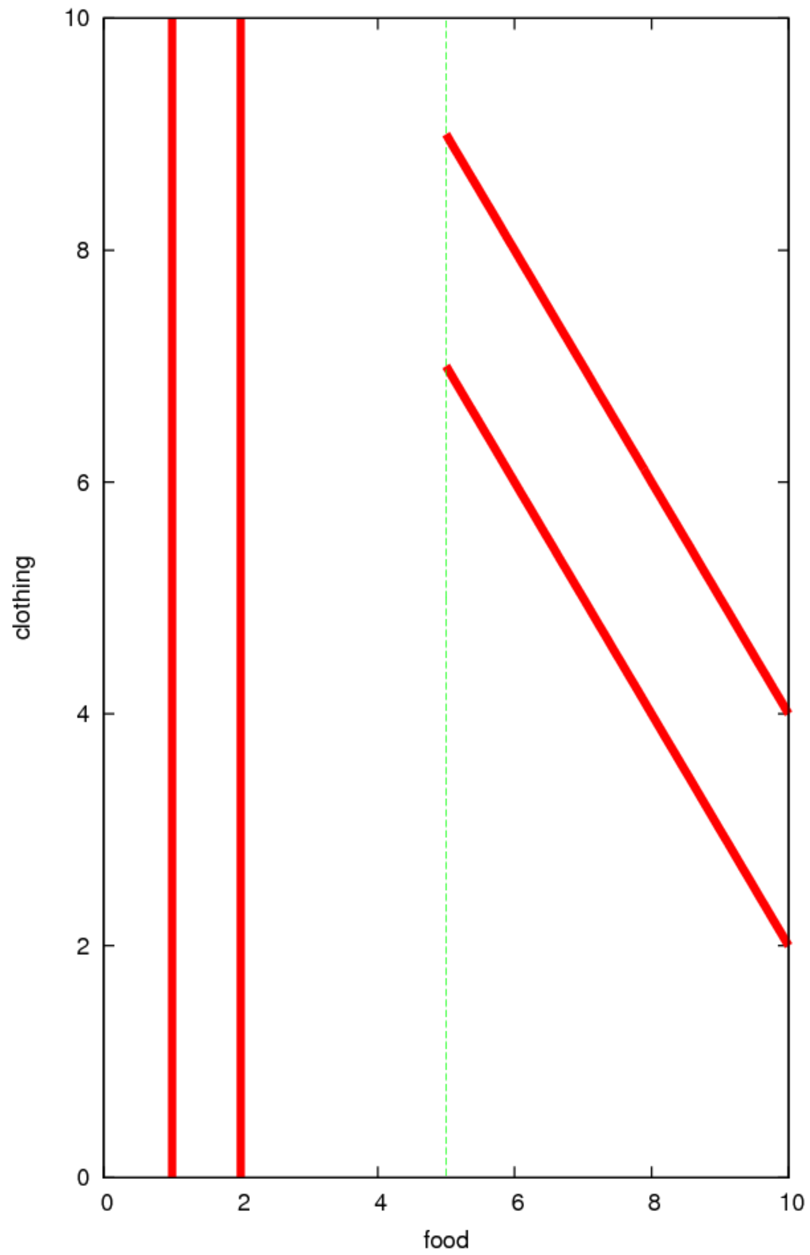

Figure (Question 1) : the at–least–as–good–as set for $(F,C) = (2,5)$ is everything on or to the right of the red vertical line (at $F = 2$) ; the at–least–as–good–as set for $(F,C) = (6,8)$ is everything on or to the right of the red diagonal line $F + C = 14$ that is also to the right of the dotted green vertical line $F = 5$

Q2. Are the preferences represented by the utility function below strictly monotonic? Convex? Explain briefly.

$$U(x_1, x_2, x_3) = x_1 x_2 - \frac{x_1 x_2}{x_3 + 1}$$

A2. Here

$$U(x_1, x_2, x_3) = x_1 x_2 (1 - \frac{1}{x_3 + 1}) = \frac{x_1 x_2 x_3}{x_3 + 1}$$

Whenever $(x_1, x_2, x_3) >> 0$, differentiation shows that the partial derivatives U_1, U_2 and U_3 are all positive. If x_1 or x_2 or x_3 equals zero, then $U = 0$, and if $(x_1, x_2, x_3) >> 0$ then $U(x_1, x_2, x_3) > 0$, so the utility function is strictly monotonic.

To check convexity of preferences, it is easiest here to transform the utility function, by setting

$$u(\mathbf{x}) \equiv \ln(U(\mathbf{x})) = \ln x_1 + \ln x_2 + \ln x_3 - \ln(x_3 + 1)$$

Then

$$u_1 = 1/x_1 \quad ; \quad u_2 = 1/x_2 \quad ; \quad u_3 = \frac{1}{x_3(x_3 + 1)} \tag{2 – 1}$$

The matrix H of second derivatives of the transformed utility function $u(\mathbf{x})$ is

$$H = \begin{pmatrix} -\frac{1}{(x_1)^2} & 0 & 0 \\ 0 & -\frac{1}{(x_2)^2} & 0 \\ 0 & 0 & -\frac{2x_3+1}{[x_3(x_3+1)]^2} \end{pmatrix}$$

This matrix has 0's off the diagonal, and negative numbers on the diagonal. So the matrix H is negative definite, which means that the function $u(\mathbf{x})$ must be concave, implying that it must also be quasi–concave, so that the preferences represented by the utility function $u(\mathbf{x})$ (or by the utility function $U(\mathbf{x})$) must be convex.

Q3. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = x_1(\sqrt{x_2} + \sqrt{x_3})$$

A3. The first–order conditions for maximization of utility, subject to the budget constraint are

$$u_1 = \sqrt{x_2} + \sqrt{x_3} = \lambda p_1 \tag{3 – 1}$$

$$u_2 = \frac{1}{2} \frac{x_1}{\sqrt{x_2}} = \lambda p_2 \tag{3 – 2}$$

$$u_3 = \frac{1}{2}\frac{x_1}{\sqrt{x_3}} = \lambda p_3 \qquad (3-3)$$

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = y \qquad (3-4)$$

Dividing equation $(3-2)$ by equation $(3-3)$ yields

$$\frac{\sqrt{x_3}}{\sqrt{x_2}} = \frac{p_2}{p_3} \qquad (3-5)$$

or

$$\sqrt{x_3} = \frac{p_2}{p_3}\sqrt{x_2} \qquad (3-6)$$

Substituting for $\sqrt{x_3}$ from $(3-6)$ into $(3-1)$ yields

$$\sqrt{x_2} + \frac{p_2}{p_3}\sqrt{x_2} = \lambda p_1 \qquad (3-7)$$

which can be written

$$\lambda = \frac{p_2 + p_3}{p_1 p_3}\sqrt{x_2} \qquad (3-8)$$

Now substitute from $(3-8)$ into $(3-2)$ to get rid of λ :

$$\frac{1}{2}\frac{x_1}{\sqrt{x_2}} = \frac{p_2}{p_1 p_3}(p_2 + p_3)\sqrt{x_2} \qquad (3-9)$$

or

$$x_1 = 2\frac{p_2}{p_1 p_3}(p_2 + p_3)x_2 \qquad (3-10)$$

Equation $(3-6)$ also implies that

$$x_3 = \frac{(p_2)^2}{(p_3)^2}x_2 \qquad (3-11)$$

so that substituting for x_1 and x_3 from $(3-10)$ and $(3-11)$ into the budget line equation $(3-4)$ gives an expression for x_2, the price and income :

$$2\frac{p_2}{p_3}(p_2 + p_3)x_2 + p_2 x_2 + \frac{(p_2)^2}{p_3}x_2 = y \qquad (3-12)$$

Multiplying both sides of $(3-12)$ by p_3 yields

$$[2p_2(p_2 + p_3) + p_2 p_3 + (p_2)^2]x_2 = p_3 y \qquad (3-13)$$

or

$$x_2 = \frac{p_3}{p_2}\frac{y}{3(p_2 + p_3)} \qquad (3-14)$$

Equation $(3-14)$ is the Marshallian demand function for good 2, since it expresses quantity demanded of good 2 as a function of income and prices.

Once the Marshallian demand function for good 2 has been derived, the Marshallian demand functions for the other goods can be found from substitution into equations $(3-10)$ and $(3-11)$. Substituting from $(3-14)$ for x_2 into $(3-10)$ yields

$$x_1 = \frac{2}{3}\frac{y}{p_1} \tag{3 - 15}$$

and substitution from $(3-14)$ for x_2 into $(3-11)$ yields

$$x_3 = \frac{p_2}{p_3}\frac{y}{3(p_2+p_3)} \tag{3 - 16}$$

(You can check that equations $(3-14)$–$(3-16)$ satisfy several sensible properties : expenditures on all 3 goods adds up to total income y, and each demand is homogeneous of degree 0 in prices and income together.)

Q4. For what values of income y and prices (p_1, p_2, p_3) will a person demand strictly positive quantities of all 3 goods, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = x_1 + 2\sqrt{x_2} + \ln x_3 \qquad ?$$

A4. Just taking the first–order conditions for an optimum, we get the conditions for the consumer's choice of consumption (x_1, x_2, x_3) subject to her budget constraint — provided it's an "interior" optimum, at which the person chooses positive quantities of each good :

$$u_1 = 1 = \lambda p_1 \tag{4 - 1}$$

$$u_2 = \frac{1}{\sqrt{x_2}} = \lambda p_2 \tag{4 - 2}$$

$$u_3 = \frac{1}{x_3} = \lambda p_3 \tag{4 - 3}$$

Substituting for λ from $(4-1)$ into $(4-2)$ and $(4-3)$ gives the Marshallian demands for goods 2 and 3 : it's pretty simple here because preferences are quasi–linear.

$$x_2 = (\frac{p_1}{p_2})^2 \tag{4 - 4}$$

$$x_3 = \frac{p_1}{p_3} \tag{4 - 5}$$

From $(4-4)$ and $(4-5)$, total expenditure on goods 2 and 3 together is

$$p_2x_2 + p_3x_3 = \frac{(p_1)^2}{p_2} + p_1 \tag{4 - 6}$$

so that the money left to spend on good 1 is

$$y - p_2x_2 - p_3x_3 = y - p_1\frac{p_1 + p_2}{p_2} \qquad (4-7)$$

Equation $(4-7)$ says that the quantity demanded of good x_1 will be positive if and only if

$$y > \frac{p_1}{p_2}(p_1 + p_2) \qquad (4-8)$$

Equation $(4-8)$ is the condition for demand for good 1 to be positive.

If income were so low that condition $(4-8)$ did not hold, then the consumer would want to spend all her money on goods 2 and 3. Maximization of $2\sqrt{x_2} + \ln x_3$ subject to the budget condition $p_2x_2 + p_3x_3 = y$ has first order conditions

$$\frac{1}{\sqrt{x_2}} = \mu p_2 \qquad (4-9)$$

$$\frac{1}{x_3} = \mu p_1 \qquad (4-10)$$

(With a little work), equations $(4-9)$ and $(4-10)$ can be reduced to

$$p_3x_3 = p_2[1 + \frac{\sqrt{1 + 4y/p_2}}{2}] \qquad (4-10)$$

If x_3 obeys equation $(4-10)$, then $p_3x_3 > p_1$ if and only if $y > \frac{p_1}{p_2}(p_1 + p_2)$. That means that if the consumer set $x_1 = 0$, and x_2 and x_3 to satisfy conditions $(4-9)$ and $(4-10)$, then $MU_3/MU_1 = 1/x_3 > p_3/p_1$ if and only if $y < \frac{p_1}{p_2}(p_1 + p_2)$.

So if $(4-8)$ did NOT hold, then the consumer's optimization involves a corner solution, at which $x_1 = 0$, $MU_3/MU_2 = p_3/p_2$ and $MU_3/MU_1 > p_3/p_1$.

Q5. Calculate the Marshallian demand functions for a consumer whose preferences can be represented by the utility function

$$u(x_1, x_2) = \frac{x_1}{1 + (1/x_2)}$$

A5. Since

$$\frac{x_1}{1 + (1/x_2)} = \frac{x_1x_2}{1 + x_2}$$

therefore

$$U(x) \equiv \ln(u(x)) = \ln x_1 + \ln x_2 - \ln(1 + x_2)$$

represents the same preferences as $u(x)$, since it is an increasing monotonic transformation of $u(x)$.

The first–order conditions for the consumer's maximization of $U(x)$ are

$$U_1 = \frac{1}{x_1} = \lambda p_1 \qquad (5-1)$$

$$U_2 = \frac{1}{x_2} - \frac{1}{(1+x_2)} = \frac{1}{x_2(1+x_2)} = \lambda p_2 \tag{5 – 2}$$

Substituting for λ from $(5-2)$ into $(5-1)$ yields

$$x_1 = \frac{p_2}{p_1} x_2(1+x_2) \tag{5 – 3}$$

Substituting for x_1 from $(5-3)$ into the budget constraint $p_1x_1 + p_2x_2 = y$ yields

$$p_2x_2 + p_2x_2(1+x_2) = y$$

or

$$p_2(x_2)^2 + 2p_2x_2 - y = 0 \tag{5 – 4}$$

Equation $(5-4)$ is a quadratic equation, which can be solved using the quadratic formula

$$x_2 = \sqrt{1 + \frac{y}{p_2}} - 1 \tag{5 – 5}$$

Since

$$x_1 = \frac{y}{p_1} - \frac{p_2x_2}{p_1}$$

therefore the Marshallian demand function for good 1 is

$$x_1 = \frac{y}{p_1} + \frac{p_2}{p_1} - \frac{1}{p_1}\sqrt{p_2y + (p_2)^2} \tag{5 – 6}$$

(It can be checked that the functions $(5-5)$ and $(5-6)$ are homogeneous of degree zero in p_1, p_2 and y together. Also, the quantities defined by $(5-5)$ and $(5-6)$ are positive whenever prices and income are positive ; the fact that the indifference curves for this person do not cross the axes means that she will never choose a corner solution.)