GS/ECON 5010 Answers to Assignment 2 October 2010

Q1. Calculate a person's Hicksian demand functions, if her expenditure function were

$$e(p_1, p_2, p_3, u) = 2(\sqrt{p_1} + \sqrt{p_2})\sqrt{p_3}u$$

A1. Shephard's Lemma is the easiest way to go here : the Hicksian demand functions are the first partial derivatives of the expenditure function. So

$$x_1^H(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_1} = \frac{\sqrt{p_3}}{\sqrt{p_1}}u \qquad (1-1)$$

$$x_2^H(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_2} = \frac{\sqrt{p_3}}{\sqrt{p_2}}u \qquad (1-2)$$

$$x_3^H(\mathbf{p}, u) = \frac{\partial e(\mathbf{p}, u)}{\partial p_3} = \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_3}}u \qquad (1-3)$$

(You can check that these demand functions are all homogeneous of degree 0 in prices ; that $p_1 x_1^H(\mathbf{p}, u) + p_2 x_2^H(\mathbf{p}, u) + p_3 x_3^H(\mathbf{p}, u) = e(\mathbf{p}, u)$; and that the matrix of derivatives of these Hicksian demand functions is negative semi–definite.)

Q2. Calculate a person's Marshallian demand functions, if her expenditure function were

$$e(p_1, p_2, p_3, u) = 2(\sqrt{p_1} + \sqrt{p_2})\sqrt{p_3}u$$

A2. Probably the easiest way to calculate the Marshallian demand functions here is to use the duality between the expenditure and indirect utility functions, and then Roy's identity.

Since $e(\mathbf{p}, v(\mathbf{p}, y)) = y$, here

$$2(\sqrt{p_1} + \sqrt{p_2})(\sqrt{p_3})v(\mathbf{p}, y) = y \qquad (2-1)$$

so that

$$v(\mathbf{p}, y) = \frac{y}{2(\sqrt{p_1} + \sqrt{p_2})\sqrt{p_3}} \qquad (2-2)$$

Differentiating $(2-2)$,

$$\frac{\partial v}{\partial y} = \frac{1}{2(\sqrt{p_1} + \sqrt{p_2})\sqrt{p_3}} \qquad (2-3)$$

$$\frac{\partial v}{\partial p_1} = -\frac{1}{2\sqrt{p_1}(\sqrt{p_1} + \sqrt{p_2})} \frac{y}{2(\sqrt{p_1} + \sqrt{p_2})\sqrt{p_3}} \qquad (2-4)$$

$$\frac{\partial v}{\partial p_2} = -\frac{1}{2\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} \frac{y}{2(\sqrt{p_1} + \sqrt{p_2})\sqrt{p_3}} \qquad (2-5)$$

$$\frac{\partial v}{\partial p_3} = -\frac{1}{2}\frac{1}{p_3}\frac{y}{2(\sqrt{p_1}+\sqrt{p_2})\sqrt{p_3}} \tag{2 – 6}$$

Roy's identity says that the Marshallian demand for good i is the partial derivative of the indirect utility function with respect to the price of good i, divided by partial derivative of the indirect utility function with respect to income — all with a minus sign in front.

So (using equations $(2-4)$ and $(2-3)$),

$$x_1^M(\mathbf{p}, y) = -\frac{\partial v/\partial p_1}{\partial v/\partial y} = \frac{y}{2\sqrt{p_1}(\sqrt{p_1}+\sqrt{p_2})} \tag{2 – 7}$$

Similarly

$$x_2^M(\mathbf{p}, y) = -\frac{\partial v/\partial p_2}{\partial v/\partial y} = \frac{y}{2\sqrt{p_2}(\sqrt{p_1}+\sqrt{p_2})} \tag{2 – 8}$$

and

$$x_3^M(\mathbf{p}, y) = -\frac{\partial v/\partial p_3}{\partial v/\partial y} = \frac{y}{2p_3} \tag{2 – 9}$$

are the Marshallian demand functions for the other 2 goods.

Q3. Is it possible that

$$e(p_1, p_2, p_3, u) = [p_3(\ln p_1 - \ln p_2) + p_2]u$$

is an expenditure function for some consumer (if $p_1 > p_2 > p_3$)?

Explain.

A3. Theorem 1.7 on page 36 of Jehle and Reny lists 7 properties which an expenditure function must satisfy.

The function defined in the question is continuous. Its first derivatives with respect to the prices are

$$e_1(\mathbf{p}, u) = \frac{p_3}{p_1}u \tag{3 – 1}$$

$$e_2(\mathbf{p}, u) = [1 - \frac{p_3}{p_2}]u \tag{3 – 2}$$

$$e_3(\mathbf{p}, u) = (\ln p_1 - \ln p_2)u \tag{3 – 3}$$

so that $e_1 > 0$, $e_2 > 0$ when $p_2 > p_3$, and $e_3 > 0$ when $p_1 > p_2$. Therefore the function is increasing in $\mathbf{p}$ whenever $p_1 > p_2 > p_3$.

Since the function can be written

$$e(\mathbf{p}, u) = p_3 \ln\frac{p_1}{p_2}u + p_2 u \tag{3 – 4}$$

, it is homogeneous of degree 1 in prices.

An expenditure function must also be concave. To check concavity, take the matrix of second derivatives, which must be negative semi–definite. From equations $(3-1)$–$(3-3)$, this matrix is

$$\begin{pmatrix} \frac{-p_3}{(p_1)^2} & 0 & \frac{1}{p_1} \\ 0 & \frac{p_3}{(p_2)^2} & -\frac{1}{p_2} \\ \frac{1}{p_1} & -\frac{1}{p_2} & 0 \end{pmatrix} u$$

This matrix is not negative semi–definite. It has a positive element on the diagonal (in the second row).

To be an expenditure function, the function must satisfy all of the properties of Theorem 1.7. Since this function is not concave, it cannot be an expenditure function.

Q4. The following table lists the prices of 3 goods, and the quantities a consumer chose of the goods, in 4 different years.

From these data, what can be concluded about how well off the consumer was in the different years? Explain briefly.

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	1	1	1	8	5	7
2	2	1	4	3	10	5
3	2	4	4	5	8	5
4	5	2	6	4	5	10

A4. This is a question about revealed preference. A person reveals that she prefers the bundle she chose in period t to some other bundle $\mathbf{x}'$ if $\mathbf{p}^t \cdot \mathbf{x}^t \geq \mathbf{p}^t \cdot \mathbf{x}'$, if $\mathbf{p}^t$ is the vector of prices in period t, and $\mathbf{x}^t$ the bundle she actually chose in period t.

To see what bundles $\mathbf{x}^t$ is revealed preferred to, the cost of all bundles must be calculated using year–t prices.

The matrix below shows the cost of all the bundles, in all of the years : the 3rd row, for example, shows the costs of all 4 bundles using year–3 prices.

If the diagonal element in row t of the matrix is at least as big as the entry in the j-th column of row t, that means that $\mathbf{x}^t$ is revealed preferred to $\mathbf{x}^j$.

20	18	18	19
49	36	38	53
64	66	62	68
92	65	71	90

There are only two rows in which the diagonal element is at least as big as some other element : rows 1 and 4. In row 1, $\mathbf{x}^1$ is more expensive than any of the other three bundles. That means the person could have afforded any one of $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3, \mathbf{x}^4\}$ in year 1 ; the fact that she chose bundle $\mathbf{x}^1$ reveals that she prefers this bundle (strictly) to any of the other 3 bundles.

In row 4, the bundle $\mathbf{x}^4$ chosen in that year is more expensive (using period 4 prices) than bundles $\mathbf{x}^2$ or $\mathbf{x}^3$. So $\mathbf{x}^4$ is revealed preferred to bundles 2 and 3.

There are no violations of *WARP* (or of *SARP*) here. The person's preferences are consistent. Her behaviour reveals that she ranks $\mathbf{x}^1$ as the best of the four bundles, and the bundle $\mathbf{x}^4$ as the next–best. But we cannot infer how she ranks bundles 2 and 3. She regards these bundles as inferior to bundles $\mathbf{x}^1$ and $\mathbf{x}^4$. But since she could not afford bundle 3 in period 2, and could not afford bundle 2 in period 3, we cannot tell which of these two bundles she prefers.

Q5. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at three different times, and the quantities x^t of the 3 goods chosen at the three different times. (For example, the second row indicates that the consumer chose the bundle $\mathbf{x} = (15, 15, 20)$ when the price vector was $\mathbf{p} = (15, 10, 10)$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	10	5	5	20	20	10
2	15	10	10	15	15	20
3	5	10	5	10	30	15
4	5	5	10	16	16	16

A5. As in the previous question, the answer here starts with the calculation of the cost of each of the 4 bundles, in each of the 4 periods. This time the matrix of those costs (with the element in row j, column i being the cost of bundle $\mathbf{x}^i$ using prices $\mathbf{p}^j$) is

350	325	325	320
600	575	600	560
350	325	425	320
300	350	350	320

In period 1, the bundle actually chosen, $\mathbf{x}^1$, is more expensive than any of the other 3 bundles. So the year 1 data shows that $\mathbf{x}^1$ is preferred (directly) to any of the other bundles. In period 2, $\mathbf{x}^2$ is revealed (directly) preferred to bundle $\mathbf{x}^4$. In period 3, the bundle actually chosen, $\mathbf{x}^3$, is more expensive than any of the other 3 bundles. So the year 3 data shows that $\mathbf{x}^3$ is preferred (directly) to any of the other bundles. Finally, period 4's data show $\mathbf{x}^4$ is revealed preferred to $\mathbf{x}^1$.

There are two violations of *WARP* here, the pairs of bundles $(\mathbf{x}^1, \mathbf{x}^3)$, and $(\mathbf{x}^1, \mathbf{x}^4)$.

But there are still more violations of *SARP*. For example, $\mathbf{x}^1$ is revealed directly preferred to $\mathbf{x}^3$, which is revealed directly preferred to $\mathbf{x}^2$, which is revealed directly preferred to $\mathbf{x}^4$, which is revealed directly preferred to $\mathbf{x}^1$.

The fact that there is a cycle of length 4 here, containing all 4 bundles, means that **every** comparison of any two bundles will violate *SARP*. There are six such comparisons : 1 versus 2, 1 versus 3, 1 versus 4, 2 versus 3, 2 versus 4, and 3 versus 4. And for each of them, a cycle can be constructed which violates *SARP*. Why? Because the cycle in the paragraph shows that **any** bundle $\mathbf{x}^i$ is indirectly revealed preferred to any other bundle $\mathbf{x}^j$.

[As an example, take bundles 2 and 3. Bundle $\mathbf{x}^3$ is directly revealed preferred to bundle $\mathbf{x}^2$. But bundle $\mathbf{x}^2$ is directly revealed preferred to bundle $\mathbf{x}^{\mathbf{4}}$ which is directly revealed preferred to bundle $\mathbf{x}^1$ which is directly revealed preferred to bundle $\mathbf{x}^3$. So bundle $\mathbf{x}^2$ is indirectly revealed preferred to bundle $\mathbf{x}^3$, and the bundles chosen in periods 2 and 3 violate *SARP*.]